RELATIVE C*-SUMS

S. K. BERBERIAN

ABSTRACT. A "bounded choice" theorem is proved for AW*-subalgebras of a C*-sum.

A classical tool in the structure theory of AW*-algebras is the C*-sum: If $A$ is an AW*-algebra and $(h_i)$ is an orthogonal family of central projections with supremum 1, then $A$ is *-isomorphic to $\bigoplus h_i A$ (the C*-algebra of all norm-bounded families $x = (x_i)$ with $x_i \in h_i A$, $\|x\| = \sup \|x_i\|$, $x^* = (x_i^*)$ and the coordinatewise algebra operations) [5, Lemma 2.5]. The following generalization has useful applications in the structure theory [2] and embedding theory [3] of AW*-algebras, the idea being that one can simultaneously decompose any AW*-subalgebra $B$ of $A$ (even if $B$ does not contain the $h_i$):

**Theorem.** Let $A$ be an AW*-algebra, $(h_i)_{i \in I}$ an orthogonal family of central projections of $A$ with $\sup h_i = 1$, $B$ an AW*-subalgebra of $A$, $e$ the unity element of $B$. Then:

1. For each $i \in I$, $h_i B$ is an AW*-subalgebra of $A$;
2. $B \subseteq \bigoplus h_i B \subseteq A$, the inclusions being AW*-embeddings;
3. $\bigoplus h_i B$ is the set of all families $(h_i, b_i)_{i \in I}$ with $b_i \in B$ and $\sup \|b_i\| < \infty$;
4. $B = \bigoplus h_i B$ if and only if $h_i e \in B$ for all $i \in I$.

**Proof.** The hypothesis on $B$ is that it is a norm-closed *-subalgebra of $A$ such that (i) $B$ contains the supremum (as calculated in $A$) of every orthogonal family of projections in $B$, and (ii) $b \in B$ implies $RP(b) \in B$ (where $RP$ denotes right-projection as calculated in $A$). It follows that (iii) $B$ is itself an AW*-algebra, and the word "orthogonal" can be omitted in (i). {For a closed *-subalgebra $B$ of $A$, the conditions (i), (ii) are equivalent to the conditions (i), (iii) [6, Lemma 2].}

(1) Let $h$ be any central projection of $A$. The mapping $B \rightarrow h A$ defined by $b \mapsto h b$ is a *-homomorphism, so its range $h B$ is a C*-algebra [4, 1.8.3]. If $x \in h B$, say $x = h b$ with $b \in B$, then $RP(x) = h RP(b)$; by hypothesis $RP(b) \in B$, so $RP(x) \in h B$. In particular, every projection $g \in h B$ may be written $g = h f$ with $f$ a projection of $B$. If $(f_i)$ is a family of projections of $B$ and if $f = \sup f_i$ (as calculated in $B$ or in $A$—it is the same), then $\sup (h f_i) = h f$; this shows that $h B$ contains the supremum (in $A$) of every family of its projections. Thus $h B$ is an AW*-subalgebra of $A$.

(2) The inclusions are obvious (on identifying $A$ with $\bigoplus h_i A$). Since $h B$ is an AW*-subalgebra of $A$, hence of $h_i A$, it follows that $\bigoplus h_i B$ is an AW*-subalgebra of $\bigoplus h_i A$.

Received by the editors November 29, 1982.
1980 Mathematics Subject Classification. Primary 46L10.

©1983 American Mathematical Society
0002-9939/82/0000-2270/$01.50
\( \bigoplus h_i A = A \). Then \( B \), being an \( \text{AW}^* \)-subalgebra of \( A \), is also an \( \text{AW}^* \)-subalgebra of \( \bigoplus h_i B \).

(3) Write \( \overline{B} = \bigoplus B_i \), where \( B_i = B \) for all \( i \in I \). The mapping \( \varphi: \overline{B} \to \bigoplus h_i B \) defined by \( \varphi((b_i)) = (h_i b_i) \) is a \(*\)-homomorphism, therefore its range \( \varphi(\overline{B}) \) is a \( C^* \)-subalgebra of \( \bigoplus h_i B \). Let us show that \( \varphi(\overline{B}) \) contains every projection \( e \) of \( \bigoplus h_i B \). Say \( e = (h_i b_i) \), where \( b_i \in B \) and \( \sup \| h_i b_i \| < \infty \). The elements \( h_i b_i \) are themselves projections, therefore \( h_i b_i = \text{RP}(h_i b_i) = h_i \text{RP}(b_i) \), where \( \text{RP}(b_i) \in B \); thus \( e = (h_i f_i) \), where the \( f_i \) are projections in \( B \), consequently \( (f_i) \in \overline{B} \) and \( \varphi((f_i)) = e \). Since the \( \text{AW}^* \)-algebra \( \bigoplus h_i B \) is generated by its projections, it follows that \( \varphi(\overline{B}) = \bigoplus h_i B \), which proves (3).

(4) If \( h_i e \in B \) for all \( i \in I \) then \( (h_i e) \) is a central partition of the unity element in the \( \text{AW}^* \)-algebra \( B \), therefore \( B = \bigoplus h_i B \). Conversely, if \( B = \bigoplus h_i B \) then \( h_i e \in h_i B \subset B \) for all \( i \in I \).

Property (3) seems striking when expressed in elementary terms: If \( a \) is an element of \( A \) such that, for every \( i \in I \), there exists \( b_i \in B \) with \( h_i a = h_i b_i \), then the family \( (b_i)_{i \in I} \) can be chosen so that \( \sup \| b_i \| < \infty \).

**References**


Department of Mathematics, University of Texas, Austin, Texas 78712