MINIMAL SURFACES WITH CONSTANT CURVATURE IN 4-DIMENSIONAL SPACE FORMS

Dedicated to Professor S. Sasaki on his 70th birthday

KATSUEI KENMOTSU

Abstract. We classify minimal surfaces with constant Gaussian curvature in a 4-dimensional space form without any global assumption. As a corollary of the main theorem, we show there is no isometric minimal immersion of a surface with constant negative Gaussian curvature into the unit 4-sphere even locally. This gives a partial answer to a problem proposed by S. T. Yau.

1. Introduction. Let $M^m(c)$ be a connected Riemannian $m$-manifold of constant sectional curvature $c$. In this paper we classify isometric minimal immersions of $M^2(K)$ into $M^4(c)$ without any global assumption and give some partial results for a classification of minimal immersions of $M^2(K)$ into $M^2(c)$.

Let $S^n(1)$ be the unit sphere in the $(n+1)$-dimensional Euclidean space $R^{n+1}$. The Clifford surface in $S^3(1)$ and the Veronese surface in $S^4(1)$ are the best known examples of minimal surfaces in space forms of positive curvature.

By a rigidity theorem of Calabi and DoCarmo and Wallach, Chern and Barbosa [2, 5, 4, 1], and a localization theorem of Wallach [10], a minimal surface with constant positive curvature in $S^4(1)$ is locally the totally geodesic $S^2(1)$ or the Veronese surface. The main theorem of this paper is

**Theorem 1.** Let $\pi : M^2(K) \to M^4(c)$ be an isometric minimal immersion of $M^2(K)$ into $M^4(c)$. If $K = c$, then $\pi$ is totally geodesic. Otherwise, either

(a) $K = 0$, $c > 0$ and $\pi$ is a locally Clifford surface in a 3-dimensional totally geodesic submanifold $M^3(c)$ of $M^4(c)$, or

(b) $K = c/3$, $c > 0$ and $\pi$ is a locally Veronese surface in $M^4(c)$.

As a corollary of the theorem, we show there is no isometric minimal immersion of the hyperbolic 2-plane, $H^2[-1]$, into $S^4(1)$ even locally. This gives a partial answer to problem 101 proposed by S. T. Yau [12, p. 692].

We cannot apply the method used to prove Theorem 1 to higher codimensional cases directly. However, in §4 we prove a nonexistence theorem for minimal...
immersions of $H^2[-1]$ into $S^5(1)$ under an additional condition on the 2nd fundamental form. Thus we still do not know whether $H^2[-1]$ can be minimally immersed in some $S^N(1)$ ($N \geq 5$).

It is a pleasure to thank Professor Takashi Ogata, who carefully read the first draft of this work and found some mistakes in it, and also Professors Manfredo P. DoCarmo and Tilla Milnor for helpful criticisms on this work.

2. Preliminaries. Let $x : M^2(K) \rightarrow M^5(c)$ be an isometric minimal immersion of $M^2(K)$ into $M^5(c)$. Let $e_1, e_2, e_3, e_4, e_5$ be local fields of orthonormal frames in $M^5(c)$ such that, restricted to $M^2(K)$, $e_1$ and $e_2$ are tangent to $M^2(K)$. Let $w_j$, $1 \leq i, j \leq 2$, and $w_\alpha$, $3 \leq \alpha, \beta, \ldots \leq 5$, be the fields of dual frames of $e_A$, $1 \leq A, B, \ldots \leq 5$. The structure equations of $M^5(c)$ are given by $dw_A = \Sigma w_\beta \wedge w_B$, $w_{AB} + w_{BA} = 0$, and $dw_{AB} = \Sigma w_{AC} \wedge w_{CB} - cw_A \wedge w_B$. The summation is taken for repeated indices. Restricting these frames to $M^2(K)$, we have $dw_{12} = -Kw_1 \wedge w_2$ and $w_\alpha = 0$, $3 \leq \alpha \leq 5$. Exterior differentiation of $w_\alpha$ gives $h_{\alpha i} = h_{\alpha ji}$, where the $h_{\alpha i}$'s are the components of the 2nd fundamental form of $x$. By the minimality of $x$, we find $h_{a i 1} + h_{a 2 2} = 0$, $3 \leq \alpha \leq 5$. From these formulae, the Gauss equation is represented by

$$\sum_{\alpha} (h_{a i 1}^2 + h_{a 2 2}^2) = c - K \geq 0.$$ 

Therefore $K = c$ occurs only when the immersion is totally geodesic. Hereafter we assume $c > K$. That is, the vector valued 2nd fundamental form $\Sigma (h_{aij} w_i \otimes w_j) e_{\alpha}$ does not vanish at any point of $M^2(K)$.

In this paper we use the theory of higher fundamental tensors developed in part 1 of [7]. Since Ogata pointed out that the proof of the $r$th ($r \geq 3$) order Codazzi equation in [7] is not correct, we use only results for 2nd fundamental tensors from [7]. However, the main theorems in [7 and 8] are valid.

We introduce some notation used in [7]. Let

$$K_{(2)} = \sum_{\alpha} (h_{a i 1}^2 + h_{a 2 2}^2),$$

$$N_{(2)} = |\Sigma h_{ai1} e_{\alpha} \wedge \Sigma h_{ai2} e_{\alpha}|^2.$$ 

The nonnegative smooth function $N_{(2)}$ on $M^2(K)$ is the square of the area of the parallelogram generated by $\Sigma h_{ai1} e_{\alpha}$ and $\Sigma h_{ai2} e_{\alpha}$. Suppose $N_{(2)}$ is identically zero on $M^2(K)$. By a lemma of Otsuki [9, p. 96] (see also Lemma 2 in [7]), there exists a 3-dimensional totally geodesic submanifold $M^3(c)$ of $M^5(c)$ such that $x(M^2(K))$ is contained in $M^3(c)$ as a minimal surface. Then we have $c > 0$ and $K = 0$ by Chen [3, Corollary 1].

In case $N_{(2)}$ is not identically zero, the set $\Omega_2 = \{p \in M^2(K) : N_{(2)}(p) \neq 0\}$ is open in $M^2(K)$. Since $\Sigma h_{ai1} e_{\alpha}$ and $\Sigma h_{ai2} e_{\alpha}$ are linearly independent at each point $x$ of $\Omega_2$, the 2nd osculating space $T_x(2)$ is spanned by those vectors and $e_i$, $1 \leq i \leq 2$. We identify the first osculating space $T_x(1)$, $x \in M^2(K)$, with the tangent space of $M^2(K)$. Let $e_\alpha$ be local orthonormal frame fields such that $e_i$, $1 \leq i \leq 2$, and $e_\alpha$, $3 \leq \alpha \leq 4$, span $T_x(2)$, $x \in \Omega_2$. Then on $\Omega_2$ we have

$$W_{i 5} = 0.$$
By taking the exterior derivative of (2.4) and using the structure equation of $M^5(c)$, we get $w_{ij} \wedge w_{35} + w_{i4} \wedge w_{45} = 0$. This allows us to introduce the quantities $h_{5ijk}$ defined by the equations

$\sum h_{aij} w_{5j} = \sum h_{5ijk} w_k$.  

The $h_{5ijk}$'s are symmetric in the Latin indices $i, j, k$ and are called components of the 3rd fundamental form of $x_{\Omega_2}$. By (2.5) and the minimality of $x$, we get $\sum h_{5ijk} = 0$. Note that

$h_{5ijk} = h^{5jk}_i$ 

is easily verified by the definition of the covariant derivatives of $h_{aij}$, since for $\alpha \geq 3$.

$Dh_{aij} = \sum h_{aij,k} w_k = dh_{aij} + \sum h_{asj} w_{si} + \sum h_{ait} w_{tj} + \sum h_{bij} w_{ba}$.

We set

$K_{(3)} = \sum h_{5111} + h_{5112}^2$.

It is easily verified that $K_{(3)}$ is an invariant of $x$ restricted to $\Omega_2$ under the fixed decomposition of the normal bundle, i.e., $e_i$ is always considered as a normal vector orthogonal to $T_x(\Omega_2)$. We put $f_{(3)} = K_{(3)}^2 - 4N_{(2)}$ and $H_a = h_{a11} + ih_{a12}$. Then we have (cf. [7]) $f_{(3)} = \sum H_a^2$ and the Codazzi equation implies

$\big( d\sum H_a^2 + 4i\left( \sum H_a^2 \right) w_{12} \big) \wedge (w_1 - iw_2) = 0$.

Therefore, by Lemma 3 of [7]

$\Delta f_{(3)} = 8K,$

wherever $f_{(3)} \neq 0$, and we get

$\Delta f_{(3)} = 8Kf_{(3)} + |Df_{(3)}|^2/f_{(3)}$.

In general, [7] gives

$\frac{1}{4} \Delta K_{(3)} = -2N_{(2)} + KK_{(3)} + K_{(3)} + \sum_{\alpha \leq 4} (h_{a11,1}^2 + h_{a11,2}^2)$.

By (2.1), (2.2) and (2.12), we obtain

$|Dh_{311}|^2 + |Dh_{421}|^2 = 2N_{(2)} - K(c - K) - K_{(3)}$.

3. Proof of Theorem 1. We assume $x(M^2(K))$ is contained in a totally geodesic submanifold $M^4(c)$ of $M^5(c)$. Then $K_{(3)}$ is identically zero by a lemma of Otsuki [9, p. 96]. By virtue of (2.13), we have

$|Df_{(2)}|^2 = 8(c - K)^{-1}f_{(2)}^3 - 16(c - 2K)f_{(2)}^2 + 8(c - K)^2(c - 3K)f_{(2)}$.

In fact, (3.1) is trivial if $f_{(2)}$ is identically zero. We assume $f_{(2)}$ is not zero. Then there exists an orthonormal frame $e_A$ for which $h_{aij}$'s satisfy $h_{312} = h_{411} = 0$ and $h_{311} > h_{412} > 0$ on an open subset of $\Omega_2$ (cf. Wong [11, p. 480]). With respect to these frame fields we have

$K_{(2)} = h_{311}^2 + h_{421}^2 = c - K = \text{constant}; \quad N_{(2)} = h_{311}^2 h_{421}^2$. 

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use
Therefore we have, by (3.2) and (2.13),
\[
\|DN(2)^2 = 4h^{2}_{311}(h^{2}_{421} - h^{2}_{311})^2 \| Dh^{311} \|^2 \\
= \frac{4h^{2}_{421}h^{2}_{321}}{h^{2}_{311} + h^{2}_{421}} \left\{ (h^{2}_{421} + h^{2}_{311})^2 - 4h^{2}_{421}h^{2}_{311} \right\} (2N(2) - K(c - K)) \\
= 4N(2)(K^2 - 4N(2))(2N(2) - K(c - K))/(c - K).
\]
By definition of \( f(2) \) and (3.2), this implies (3.1).

Lemma 1. Let \( x : M^2(K) \rightarrow M^4(c) \) be an isometric minimal immersion with \( N(2) \neq 0 \) on \( M^2(K) \). Then we have \( K = c/3 \) and \( c > 0 \). Locally \( x \) is the Veronese surface in \( M^4(c) \).

Proof. If \( f(2) \) is identically zero on \( M^2(K) \), then \( \Sigma h_{a1} e_a \) and \( \Sigma h_{a2} e_a \) are orthogonal and have the same nonzero length (cf. [7, p. 475]). By normalizing these vectors, we adopt them as a part of the basis of \( T_x^{(2)} \), \( x \in M^2(K) \). For these new frame fields, we have \( h_{311} = h_{412} > 0 \), \( h_{312} = h_{411} = 0 \). The Gauss equation implies
\[
h_{311}^2 = h_{412}^2 = (c - K)/2 = \text{constant} > 0.
\]
We have \( Dh_{311} = dh_{311} = 0 \) and \( 2w_{12} = w_{34} \), as seen by the formulas
\[
Dh_{312} = h_{311}(2w_{12} - w_{34}) = h_{311,2}w_1 - h_{311,1}w_2 = 0.
\]
Exterior differentiation of \( 2w_{12} = w_{34} \) gives \( K = h_{311}h_{421} \). It follows from these formulas that \( K = h_{311}^2 \) is positive, and we can see \( K = c/3 \) by (2.13), which implies \( c > 0 \).

It is known that such an immersion represents locally the Veronese surface (cf. for instance [11, Theorem 4.2]).

Next we will show that the case of \( f(2) \neq 0 \) cannot happen. By (2.11) and (3.1), we obtain
\[
\Delta f(2) = 8(c - K)^{-1}f(2) - 8(2c - 5K)f(2) + 8(c - K)^2(c - 3K).
\]
If we set \( \theta = f(2) \), then (3.1) and (3.3) are expressed in the following way: \( \Delta\theta = \phi(\theta) \), \( |D\theta|^2 = f(\theta) \), where \( \phi(\theta) \) and \( f(\theta) \) are polynomials for \( \theta \) with constant coefficients. It is then proved that \( \theta \) must be a constant function. If \( \theta \) is not constant then there exist local coordinates \( (\theta, \tau) \) on \( M^2(K) \) such that the first fundamental form is
\[
ds^2 = \left( d\theta^2 + \exp \left( 2\int f^{-1}d\theta \right) d\tau^2 \right)/f(\theta)
\]
and the Gauss curvature \( K \) satisfies
\[
fK + (\phi - f'(\phi - \frac{1}{2}f')) + f(\phi' - \frac{1}{2}f'') = 0,
\]
where the prime denotes differentiation with respect to \( \theta \) (cf. Eisenhart [6, p. 164]). The left-hand side of (3.4) is a polynomial in \( \theta \) such that the coefficients of \( \theta^3 \) and \( \theta \) are \( 8(8c - 27K)/(c - K) \) and \( 8(8c - 27K)/(c - 3K) \), respectively. Since (3.4) is a nontrivial polynomial with constant coefficients, \( \theta \) must be a constant. Therefore \( f(2) \) is a nonzero constant, which implies \( K = 0 \) by (2.10) and \( f(2) = c^2 - 4N(2) \). By (3.3), we have \( f(2) = c^2 \), which is a contradiction because we have assumed \( N(2) \neq 0 \). This proves Lemma 1.
We prove Theorem 1 as follows: If $N_2$ is identically zero, then $x$ is totally geodesic or we have $K = 0$ and $c > 0$. Moreover, in this case, $x(M^2(0))$ is contained in a 3-dimensional totally geodesic $M^3(c)$ as part of a Clifford surface [3].

When $N_2$ is not identically zero on $\Omega_2$, $N_2$ is a positive constant on $\Omega_2$. Since $\Omega_2$ is open and closed, we have $\Omega_2 = M^2(K)$, because $M^2(K)$ is connected by definition. Hence Theorem 1 has been proved.

4. The 3rd fundamental form. In this section we study the case $K_3 \neq 0$, which means that the minimal immersion $x : M^2(K) \rightarrow M^5(c)$ is full. We define covariant derivation for the 3rd fundamental tensor (cf. [7]) by

$$\bar{D}h_{5ijk} = \sum h_{5ijk,l}v_i = dh_{5ijk} + \sum h_{5sjk}w_s + \sum h_{5isk}w_s + \sum h_{5jsk}w_s.$$  \hspace{1cm} (4.1)

Exterior differentiation of (2.5) gives

$$\sum h_{5ijk,;l}w_i = - \sum h_{aif,j}w_a \wedge w_i.$$  \hspace{1cm} (4.2)

When we set $w_a = \sum a_a w_j$, (4.2) is equivalent to

$$h_{5ijk,;l} - h_{5ijk,;k} = \sum h_{aif,k}a_{a} - \sum h_{aif,k}a_{a}.$$  \hspace{1cm} (4.2')

By (2.5) we have

$$\sum h_{aij}a_{ak} = h_{5ijk}.$$  \hspace{1cm} (4.3)

**Lemma 2.** Let $x : M^2(K) \rightarrow M^5(c)$ be a full isometric minimal immersion. Then the 3rd order Codazzi equations

$$h_{5ijk,;l} = h_{5ijk,;k}$$  \hspace{1cm} (4.4)

hold if $N_2$ is constant on $M^2(K)$.

**Proof.** $f_2$ is constant on $M^2(K)$. If $f_2$ is zero, then we take frame fields $e_4$ such that $h_{311} = h_{412}$ and $h_{312} = h_{411} = 0$. In the case of $f_2 \neq 0$, we take Y. C. Wong’s frame, for which $h_{311} > h_{412}, h_{312} = h_{411} = 0$ (cf. [11, p. 480]). Since $N_2$ is constant, in both cases, $h_{311}$ and $h_{421}$ are also locally constant, so $Dh_{3ij} = Dh_{4ij} = 0$. It follows that the right-hand side of (4.2') vanishes. Q.E.D.

Hereafter we assume $N_2$ is positive constant on $M^2(K)$. $f_2$ is also constant. Since we have $Dh_{aij} = 0$, by (2.13), $K_3$ is constant. We take frame fields $e_i$ locally for which $h_{5112}$ vanishes on a neighborhood of $M^2(K)$. It shall be remarked that $e_3$ and $e_4$ are any vectors such that $e_3$, $e_3$, and $e_4$ are bases of $T^i$. Since $K_3 = h_{5111} + h_{3111}$ is a nonzero constant, $h_{5111}$ is locally a nonzero constant. From the formulas

$$Dh_{5111} = 0$$

and

$$Dh_{5112} = 3h_{5111}w_2 = h_{5111,1}w_1 - h_{5111,1}w_2 = 0,$$

which is derived by Lemma 2, we get $w_2 = 0$. It implies $K = 0$ so $c$ is positive by (2.1). Therefore Lemma 2 in [8] holds, so $x$ is locally a generalized Clifford surface of $M^5(c)$ described in [8]. Summarizing, we get this result.

**Theorem 2.** Let $x : M^2(K) \rightarrow M^5(c)$ be a full isometric minimal immersion with $N_2$ constant on $M^2(K)$. Then $K = 0$ and $c$ is positive. Locally $x$ is one of the generalized Clifford surfaces described in [8].
Remarks. (1) If the $r$th ($\geq 3$) fundamental tensors all satisfy the $r$th order Codazzi equation, then we can prove a local classification theorem for minimal immersions $M^2(K) \to M^m(c)$ by methods similar to those proving Theorem 1.

(2) Let $x : M^2(K) \to M^5(c)$ be any full isometric minimal immersion. Then by (2.13) we have $|Dh_{311}|^2 + |Dh_{421}|^2 \leq 2N_{(2)} - K(c - K)$. It follows that

\begin{align}
(4.5) \quad |Df_{(2)}|^2 &\leq 8(c - K)^{-1}f_3^2 - 16(c - 2K)f_4^2 + 8(c - K)^2(c - 3K)f_{(2)}, \\
(4.6) \quad \Delta f_{(2)} &\leq 8(c - K)^{-1}f_2^2 - 8(c - 2K)f_4^2 + 8(c - K)^2(c - 3K).
\end{align}

Under suitable assumptions on the topology of $M^2(K)$, the author conjectures that the $f_{(2)}$ must be constant.

References


Department of Mathematics, College of General Education, Tōhoku University, Kawauchi, Sendai, 980, Japan