A NOTE ON JENSEN'S COVERING LEMMA

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Abstract. We show that Jensen's covering lemma does not hold for order-types instead of cardinalities.

A famous result of Jensen says that if \(0^\dagger\) does not exist then every uncountable set can be covered by a constructible set of the same power \([1]\). We show that an iterated forcing construction can produce a model such that for every \(\alpha < \omega_2\) there is a set (in the enlarged model) of order-type \(\omega_1\) which cannot be covered by an old set of order-type \(\alpha\).

Theorem. Assume that \(V\) is a countable, transitive model of ZFC + GCH. Then there is a cardinal preserving generic extension \(V[G]\) such that for every \(\alpha < \omega_2\) there is an \(X \subseteq \omega_2\), \(\text{tp}(X) = \omega_1\), such that for \(Y \in V\) with \(Y \supseteq X\), \(\text{tp}(Y) < \alpha_1\), does not exist.

Proof. The appropriate notion of forcing will be \(P_\omega\), where \(P_\alpha(\alpha \leq \omega_2)\) is defined by induction. \(P_0\) is just the trivial notion of forcing. \(P_{\alpha+1} = P_\alpha \ast Q_\alpha\), where \(Q_\alpha\) will be defined later; for limit \(\alpha\) we take inverse limits when \(\text{cf}(\alpha) = \omega\) and direct limits if \(\text{cf}(\alpha) > \omega\).

We define \(Q_\alpha\) (inside \(V^{P_\alpha}\)) as follows: \((f, A) \in Q_\alpha\) if and only if

1. \(\text{Dom}(f) \subseteq \omega_1\),
2. \(\omega_1^{\alpha+1} \xi < f(\xi) < \omega_1^{\alpha+1}(\xi + 1)\) \((\xi \in \text{Dom}(f))\),
and
3. \(\text{tp}(A \cap [\omega_1^{\alpha+1} \xi, \omega_1^{\alpha+1}(\xi + 1)]) < \omega_1^{\alpha+1}\).

The partial order is defined by \((f', A') \leq (f, A)\) iff \(f' \supseteq f\), \(A' \supseteq A\), and \(f'(\xi) \notin A\) for \(\xi \in \text{Dom}(f') - \text{Dom}(f)\). It is easy to see that this relation is transitive.

In order to prove that forcing with any of the \(P_\alpha(\alpha \leq \omega_2)\) cardinals and cardinal arithmetic remain we need some tools elaborated by Baumgartner \([2]\). A partial order \(P\) is \(\sigma\)-closed if every decreasing sequence \(p_0 \geq p_1 \geq \cdots \geq p_n \geq \cdots (n < \omega)\) has a lower bound. \(P\) is well-met iff every two compatible elements have a greatest lower bound. \(P\) is \(\omega_1\)-linked iff it can be written as \(\bigcup \{R_\tau : \tau < \omega_1\}\) where the elements in any of the \(R_\tau\)'s are pairwise compatible. Baumgartner proves that if \(P_\alpha(\alpha \leq \omega_2)\) is defined as above and \(Q_\alpha\) is (in \(V^{P_\alpha}\)) \(\sigma\)-closed, well-met and \(\omega_1\)-linked, then \(P_\omega\) is \(\sigma\)-closed and has the \(\omega_2\)-chain condition.

Lemma 1. The partial ordering \(Q_\alpha\) is \(\sigma\)-closed, well-met and \(\omega_1\)-linked.
Proof. Assume \((f_n, A_n)\) is decreasing. At least, \(f = \bigcup f_n\), \(A = \bigcup A_n\) seems to give a candidate. As \(\omega^{\alpha+1}_1\) is indecomposable into countably many smaller ordinals, the \((f, A)\) is a condition. Assume \(\xi \in \text{Dom}(f) - \text{Dom}(f_n)\), and there is an \(m\) with \(\xi \in \text{Dom}(f_{m+1}) - \text{Dom}(f_m)\). Then \(\xi \notin A_m \supseteq A_n\).

The ordering is \(\omega_1\)-linked as \((f, A)\) and \((f, B)\) are always compatible and by CH there are only \(\omega_1\) first coordinates.

To prove that \(Q_\alpha\) is well-met assume \((f, A)\) and \((g, B)\) are compatible and \(\text{Dom}(f) < \text{Dom}(g)\). As some \((h, C) \leq (f, A), (g, B)\), so if \(\xi \in \text{Dom}(f) - \text{Dom}(g)\) surely \(\xi \in \text{Dom}(h) - \text{Dom}(f)\) so \(g(\xi) = h(\xi) \notin A\). This gives that \((f \cup g, A \cup B)\) is the g.l.b. for \((f, A), (g, B)\).

We finish the proof by showing that the major conclusion of the theorem holds in \(\mathcal{V}^P\).

Lemma 2. Forcing with \(Q_\alpha\) defines a set \(X_\alpha\) of order-type \(\omega_1\) which cannot be covered by a set of order-type \(\leq \omega^{\alpha+1}_1\) in the ground model.

Proof. Let \(G\) be a generic set over \(Q_\alpha\). Put \(X_\alpha = \bigcup \{\text{Rug}(f) : (f, A) \in G\}\). As \(D_\xi = \{(f, A) : \xi \in \text{Dom}(f)\}\) is clearly dense for \(\xi < \omega_1\), the order-type of \(X_\alpha\) is \(\omega_1\). Assume \(Y\) is the ground model \(Y \subseteq \omega^{\alpha+2}_1\), tp \(Y \leq \omega_\alpha\) and \((f, A)^{1+} X_\alpha \subseteq Y\). Then \((f, A \cup Y)\) is a condition forcing \(X_\alpha \cap Y \subseteq \text{Rug}(f)\) which is countable, so \(X_\alpha \subseteq Y\) cannot hold.

References


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