A COUNTEREXAMPLE TO
THE GENERALIZED BANACH THEOREM

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Abstract. We show that it is consistent that the family of Borel maps of class 2 differs from the family of pointwise limits of Borel maps of class 1. This gives an answer to a question raised by W. G. Fleissner.

1. Let $X, Y$ be metric spaces. For $0 < \alpha < \omega_1$, denote by $\Sigma_\alpha X$ the family of sets in $X$ of additive class $\alpha$. Define the Borel classes by

$$\psi_\alpha(X, Y) = \{ f: X \to Y : \forall G \in \Sigma_\alpha Y \; f^{-1}[G] \in \Sigma_\alpha X \}$$

and the Banach classes for $1 < \alpha < \omega_1$ by

$$\phi_\alpha^*(X, Y) = \psi_\alpha(X, Y)$$

and

$$\phi_\alpha^*(X, Y) = \text{family of all limits of pointwise convergent sequences}$$

$$\text{of maps from } \bigcup_{\beta < \alpha} \phi_\beta^*(X, Y) \quad (\alpha > 1).$$

Correspondence between the Borel classes and the Banach classes is expressed by

**Banach Theorem.** Let $Y$ be a separable metric space. Then $\phi_\alpha^*(X, Y) = \psi_\alpha(X, Y)$ or $\psi_{\alpha+1}(X, Y)$ according as $\alpha$ is finite or infinite.$^1$

Recall that the inclusion $\phi_\alpha^*(X, Y) \subseteq \psi_\alpha(X, Y)$ or $\psi_{\alpha+1}(X, Y)$, according as $\alpha$ is finite or infinite, holds for any metric space $Y$.

Recently W. G. Fleissner [F1] introduced an axiom called Proposition P. He proved that Proposition P is consistent, assuming that ZFC + 'there exists a supercompact cardinal' is consistent and that it implies, among other things, the Banach Theorem for any metric space $Y$. Later Fleissner, Hansell and Junnila [FHJ] proved that Proposition P is also implied by the Product Measure Extension Axiom.

In [F1] the author asks whether it is consistent that $\phi_\alpha^*(X, Y) \neq \psi_2(X, Y)$. In this note we show how the well-known Miller Theorem [M] can be used to give an affirmative answer to this question.

2. Recall that a $Q$ set is an uncountable subset $X$ of the real line $\mathbb{R}$ such that every subset of $X$ belongs to $\Sigma_1 X$. It follows from the Miller Theorem (cf. [F2, Theorem 23]) that the following proposition is consistent with ZFC.
PROPOSITION. There is a subset $X$ of $\mathbb{R}$ with cardinality $\omega_3$ such that every subset of $X$ belongs to $\Sigma_2 X$, but every $Q$ set contained in $X$ has cardinality $< \omega_3$.

THEOREM. If $|X|$ is the set $X$ with discrete metric, then $f \in \psi_2(|X|, X) \setminus \psi_2^*(X, |X|)$ for every injection $f$ from $X$ to $|X|$.

Proof. Clearly $f \in \psi_2(X, |X|)$. Suppose $f \notin \psi_2^*(X, |X|)$. Then there is a sequence $\{f_n; n \in \omega\} \subseteq \psi(X, |X|)$ such that $f$ is the pointwise limit of $f_n$. Put $A_n = \{x \in X: f_n(x) = f(x)\}$. Since $X = \bigcup\{A_n; n \in \omega\}$ there is $n_0 \in \omega$ such that the cardinality of $A_{n_0}$ is $\omega_3$. This implies $A_{n_0}$ is not a $Q$ set. Hence, since $f|A_{n_0}$ is an injection, $f|A_{n_0} \not\in \psi(A_{n_0}, |X|)$. On the other hand, $f|A_{n_0} \in \psi(A_{n_0}, |X|)$ and $f|A_{n_0} = f|A_{n_0}$, contradiction.

Remark 1. Similarly, Miller’s Theorem implies that for any nonlimit $\alpha > 1$ there is a set $X_\alpha$ such that $\psi_\alpha^*(X_\alpha, |X_\alpha|) \neq \psi_\alpha(X_\alpha, |X_\alpha|)$ or $\psi_{\alpha+1}(X_\alpha, |X_\alpha|)$ according as $\alpha$ is finite or infinite.

Remark 2. Fleissner proved in [F₁] that under Proposition P for every function $f$ from $\psi(X, Y)$, where $X$ and $Y$ are arbitrary metric and Banach spaces, respectively, there is a residual set $T$ in $X$ such that $f|T$ is continuous and he raised the question as to whether ‘residual’ can be replaced by ‘dense’. Observe, however, that if $f$ is a function from the rationals to $\mathbb{R}$ which has a discrete image, then $f \notin \psi(Q, R)$ and $f|T$ is not continuous for any dense subset $T$ of $Q$.

References


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