ON THE ALEXANDER POLYNOMIAL OF A CYCLICALLY PERIODIC KNOT

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Abstract. We show that a theorem of Burde on the Alexander polynomial of a cyclically periodic knot \( K \), such that \( \Delta_2(K) = 1 \), may be extended to all cyclically periodic knots.

A knot \( K \) in a homology 3-sphere \( \Sigma \) has cyclic period \( q \) if it is invariant under an automorphism of \( \Sigma \) of order \( q \) which has fixed point set a circle disjoint from \( K \). In [3] we gave new proofs of theorems of Burde [1] and Murasugi [5] on the Alexander polynomial \( \Delta_1(K) \) of such a knot.

A refinement of the second lemma of [3] enables us to remove the restriction \( \Lambda_2(F) = 1 \) from Burde’s theorem. We shall adopt the notation of [3].

Theorem. Let \( K \) be a knot in an homology 3-sphere \( \Sigma \) which has cyclic period \( q \), where \( q = p^r \) for some prime \( p \). Then either

(i) \( \Delta_q(K) \equiv 1 \) modulo \( (p) \) or

(ii) the degree of the \( q \)-th roots of unity over the splitting fields of \( \Delta_1(K) \) is at most \( n \), where \( \Delta_q(K) \neq 1 \) but \( \Delta_{n+1}(K) = 1 \).

Proof. The argument given in [3] shows that if (i) fails then the \( \mathbb{Q}[t, t^{-1}]- \)module \( C = (G'/G'') \otimes \mathbb{Q} \) (the rational homology of the infinite cyclic cover of the knot complement) has an automorphism of order exactly \( q \). (For otherwise the \((q/p)\)th power of the given periodic self-homeomorphism \( \tau \) of \( \Sigma \) induces the identity on \( G'/G'' \). Applying Milnor duality and the Wang sequence with coefficients \( \mathbb{Z}/p\mathbb{Z} \) to the \( p \)-fold branched cyclic covering map from \( \Sigma \) to \( \tilde{\Sigma} \), the results of 0-framed surgery on \( K \) in \( \Sigma \) and on the image of \( K \) in \( \Sigma/\tau(q/p) \), respectively, we conclude that \( (G'/G'') \otimes (\mathbb{Z}/p\mathbb{Z}) \) must be 0, and so \( \Delta_q(K) \equiv 1 \) modulo \( (p) \).) Then for some irreducible factor \( \delta \) of \( \Delta_q(K) \), the \( \delta \)-primary submodule of \( C \) has an automorphism \( \tau \) of order exactly \( q \), so we may assume \( \Delta_q(C) = \delta^q \). Let \( L = \mathbb{Q}[t, t^{-1}]/(\delta) \). If \( m \geq 1 \) the kernel of the natural map from \( \text{Aut}(C/\delta^{m+1}C) \) to \( \text{Aut}(C/\delta^mC) \) is isomorphic to \( \text{Hom}(C/\delta^{m+1}C, \delta^mC/\delta^{m+1}C) \) and so is an \( L \)-vector space. In particular, it is torsion free. Therefore the image of \( \tau \) in \( \text{Aut}(C/\delta C) \cong \text{GL}(n, L) \) has order exactly \( q \). Let \( A = L[\tau] \) be the subalgebra of \( M(n, L) \) generated by this element. Then \( \dim_L A \leq n \) by the Cayley-Hamilton theorem. On the other hand, the image of \( \tau \) in the semisimple algebra \( A/\text{rad} A \) again has order exactly \( q \), as the kernel of the map on

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units is again an iterated extension of $L$-vector spaces. The algebra $A/\text{rad } A$ is a product of fields containing $L$, so some factor is an extension $L'$ of $L$ which contains the $q$th roots of unity and for which $[L':L] = \dim_L L' \leq \dim_L A \leq n$. Since $L$ is contained in the splitting field for $\Delta_i(K)$, the theorem follows.

**Remarks.** Suppose (ii) holds. If $q = p^r$, with $r > 1$, then either $L = L'$, and so contains the $q$th roots of unity, or $[L':L] \geq p$, and so $p \leq n$. If $q = p$ and $L$ is linearly disjoint from the field $Q(\zeta)$ of $p$th roots of unity, that is if $L \cap Q(\zeta) = Q$, then $[L':L] = p - 1$, so $p \leq n + 1$.

If $\Delta_2(K) = 1$, so that $n = 1$, we obtain Burde's theorem [1] (itself an extension of a result of Trotter [8]). Burde's version applies to all but four of the prime knots with ten crossings or less (excepting $8_{18}, 9_{40}, 10_{99}$ and $10_{123}$), but does not usefully apply to the $n$-fold connected sum of a knot $K$ with itself, for $\Delta_2(\#^nK)$ is nontrivial unless $\Delta_1(K) = 1$ or $n = 1$.

For other recent work on knots with cyclic periods see [2, 3, 4, 6, 7]. (Note that there are some discrepancies between the tables of [4] and [6].)

**References**


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