A NOTE ON COMPACT GROUPS

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Abstract. We show that the product of certain subsets in a compact connected topological group is the group itself.

Let $G$ be a connected topological group. It is well known that $G$ is generated by any neighborhood $V$ of the identity $1$ of $G$, i.e. $G = \bigcup_{n=1}^{\infty} V^n$. If $G$ is also compact, then $G = V^k$ for some $k$. It is natural to ask whether the above statement is true if we replace the neighborhood of $1$ by some other types of subsets of $G$. The purpose of this note is to show one such possibility. Precisely, we prove:

Proposition. Let $G$ be a compact connected Hausdorff topological group and $\mu$ the (normalized) Haar measure on $G$. Let $A_1, A_2, \ldots$ be a sequence of Borel measurable sets in $G$ such that $\inf \mu(A_i) > 0$. Then $G = A_1 A_2 \cdots A_n$ for some $n$.

First, we need

Lemma. Let $A$ and $B$ be Borel subsets in $G$ with positive measure. Then $AB$ has nonvoid interior.

Proof. This is a special case of a well-known general result. For a proof, see [3,(20.17), p. 296].

Proof (of the Proposition). We may suppose by the Lemma that $A_1$ is an open subset. There are elements $a_1, a_2, \ldots$ in $G$ such that $1 \in A_1 a_1$, and $1 \in a_n^{-1} A_n a_n$, for $n > 1$. Let $B_1 = A_1 a_1$, $B_n = a_n^{-1} A_n a_n$. Then $\mu(B_n) = \mu(A_n)$. Since $A_1 A_2 \cdots A_n = B_1 B_2 \cdots B_n a_n^{-1}$, it suffices to prove the Proposition for the sequence $B_1, B_2, \ldots$ of Borel sets containing the identity element.

Let $S = \{ x \in G \mid x \in B_i$ for infinitely many indices$\}$. Then $S = \bigcap_{k=1}^{\infty} \bigcup_{i=k}^{\infty} B_i$. We note that $1 \in S$ and $\mu(S) \geq \inf \mu(B_i)$. Let $S^\ast$ be the semigroup generated by $S$, i.e. $S^\ast = \bigcup_{n=1}^{\infty} S^n$. Then $1 \in S \subseteq S^\ast$. Because $S$ has positive measure, by the Lemma we know that $S^\ast$ has nonvoid interior $W$. Now we shall prove that $1$ is in the interior $W$ of $S^\ast$. Let $x$ be any element in $W$. Since $G$ is compact, the closure of the semigroup generated by $x^{-1}$ is a compact subgroup; in particular, it contains $1$. Thus $x^{-1}$ is in the neighborhood $x^{-1}W$ of $1$ for some $n \geq 1$. Then $x^{1-n} \in W$, and $1 = x^{n-1} x^{1-n} \in x^{-1} W \subset W^n = W$. In other words, $S^\ast$ is a neighborhood of $1$. Since $G$ is connected we conclude that $G = S^\ast$ (cf. the beginning of this note). Finally, since $S^\ast$ is generated by the elements which appear infinitely often in the
sequence $B_1, B_2, \ldots, S^* \subseteq \bigcup_{n=1}^{\infty} B_1 B_2 \cdots B_n$. Using the fact that $G = S^*$ is compact and each $B_1 B_2 \cdots B_n$ is open, we conclude that $G = S^* = B_1 B_2 \cdots B_k$ for some $k$, and the proof is complete.

**Remark.** Observe that, given a sequence of open neighborhoods $V_1 V_2, \ldots$ of 1, it is not always true that $G = V_1 V_2 \cdots V_k$ for some $k$. Simple examples such as taking small intervals in the circle illustrate this fact.

For earlier works in this direction, but on abelian locally compact groups, we refer to [1, 2, 5] and references therein. This note shows certain kinds of ergodicity of subsets in compact groups. For applications of this kind, we refer to [4].

**References**


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