PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 89, Number 2, October 1983

ARCHIMEDEAN, SEMIPERFECT AND π-REGULAR LATTICE-ORDERED ALGEBRAS WITH POLYNOMIAL CONSTRAINTS ARE f-ALGEBRAS

STUART A. STEINBERG

ABSTRACT. It is shown that a lattice-ordered algebra is embeddable in a product of totally ordered algebras provided (i) it is archimedean, contains a left superunit which is an f-element, and satisfies a polynomial identity $p(x) \geqslant 0$ or $f(x, y) \geqslant 0$ (for suitable $f(x, y)$); or (ii) it is unital, and semiperfect, π-regular, or left π-regular, and some power of each element is positive.

A torsion-free lattice-ordered algebra R over the commutative unital totally ordered domain F is called an *l-algebra* if for all $r, s \in R^+ = \{r \in R\colon r \geqslant 0\}$ and $\alpha \in F^+$,

$$r \wedge s = 0 \text{ implies } \alpha r \wedge s = 0.$$

Let

$$T = \{r \in R\colon u \wedge v = 0 \text{ implies } |r|u \wedge v = u|r| \wedge v = 0\}.$$

Then T consists of the *f-elements* of R; T is a convex l-subalgebra of R which contains 1, if $1 \in R^+$; and R is an *f-algebra* precisely when $T = R$.

The variety of f-rings, which was introduced by Birkhoff and Pierce in [**1**], has been the most extensively studied class of l-rings. This is because an f-algebra is a subdirect product of a family of totally ordered algebras, and, hence, computations in f-algebras can frequently be reduced to the totally ordered case. However, larger classes and varieties of l-algebras have been studied by Birkhoff and Pierce [**1**], Diem [**2**], Shyr and Viswanathan [**3**], and Steinberg [**4–7**].

The l-algebra R is *l-prime* if the product of two of its nonzero l-ideals is nonzero, and an *l-domain* if the product of two nonzero positive elements is nonzero. R is *reduced* if $a^2 = 0$ implies $a = 0$. In [**2**, p. 79] Diem asked if an l-prime l-ring R in which the square of every element is positive must be an l-domain. In [**7**] we have shown that R must be a domain if it is unital or the left and right annihilator ideals of T vanish. More generally, the same conclusion follows if the identity $x^2 \geqslant 0$ (actually, $(x^2)^- = 0$) is replaced by more general polynomial constraints. Let $F[x, y]$ be the free noncommutative F-algebra in two variables x and y. A polynomial $f(x, y) \in F[x, y]$ is *nice* if

$$f(x, y) = -g(x, y) + p(y) + h(x, y)$$

Received by the editors July 12, 1982.
1980 *Mathematics Subject Classification*. Primary 06F25; Secondary 16A86.
Key words and phrases. Lattice-ordered ring, totally ordered ring, polynomial constraint.

where $0 \neq g(x, y)$ is of degree 1 in x and has all its coefficients positive, and $h(x, y) = 0$ or each of its monomials has degree at least 2 in x. $f(x, y)$ is *left* (*right*) *nice* if $g(x, y)$ has a monomial which begins (ends) with x, and is *k-nice* if $h(x, y) \in F[x^k, y]$. For example, $-x$ and $(x - y)^2$ are left and right 2-nice polynomials. From [7] we have the

THEOREM. *Let R be an l-prime l-algebra over the totally ordered domain F.*

(1) *If $1 \in R^+$ and if $u \wedge v = 0$ implies there is a nice polynomial $f(x, y) \in F[x, y]$ with $f(u, v) \geqslant 0$, then R is an l-domain.*

(2) *Each of the following conditions implies that R is a reduced l-domain.*

(a) *$1 \in R$ and for each invertible element $u \in R$ there is a polynomial $p(x) \in F[x]$ with $p(u) \geqslant 0$ and $0 \neq p'(1) \in R^+$ ($p'(x)$ is the derivative of $p(x)$).*

(b) *If $r \in R$ and $rT = 0$ or $Tr = 0$, then $r = 0$; and there is a right and left k-nice polynomial $f(x, y) \in F[x, y]$ (with $k \geqslant 2$) such that R satisfies $f(x, y^+)^- = 0$.*

(3) *If F is a field and $1 \in R^+$, then each of the following implies that R is a domain.*

(a) *For each $r \in R$ there is a polynomial $p(x) \in xF[x]$ with $p(r) \in R^+$, $p(1)p'(1) \neq 0$, and $p'(1) \in R^+$.*

(b) *R satisfies the identity $p(x)^+ p(x)^- = 0$ where $p(x) \in F[x]$ has only odd terms and $p(1)p'(1) \neq 0$.*

In this note we investigate l-algebras with such polynomial constraints. In particular, we show that the squares positive hypothesis in [**4** and **6**] can be relaxed; that is, l-algebras with certain constraints that are archimedean, semiperfect, algebraic, π-regular or left π-regular must be f-algebras.

If r and s are two elements of the l-algebra R, then r *is infinitely smaller than s with respect to F*, written $r \ll s$, if $\alpha|r| \leqslant |s|$ for each $\alpha \in F$. R is *archimedean over F* if $r \ll s$ implies $r = 0$. R is a *PPI l-algebra* over F if R satisfies the identity $f(x, y)^- = 0$ where $f(x, y) \in F[x, y]$ and $f(x, y) \notin F$. By a *left superunit* e in R we mean an element $e \in R^+$ such that $ex \geqslant x$ for each x in R^+. The element $a \in R$ is a *left f-element* if $b \wedge c = 0$ implies $|a|b \wedge c = 0$, and a *weak order unit* if $|a| \wedge b = 0$ implies $b = 0$. For notational convenience we note that $F[x, y]$ is an l-algebra with positive cone $F^+[x, y]$, and we will denote the positive part, negative part and absolute value of $f(x, y)$ by $f^+(x, y)$, $f^-(x, y)$ and $|f|(x, y)$, respectively.

1. Archimedean l-algebras. To show that archimedean PPI l-algebras are f-algebras we require two lemmas.

LEMMA 1. *Let $a, e \in R^+$ and suppose that there exists $p(x) = \alpha_0 + \alpha_1 x + \cdots + \alpha_n x^n \in F[x]$ of degree $n \geqslant 1$ with $\alpha_n > 0$ and $0 \leqslant p(\alpha e - a)$ for each α in a cofinal subset of F^+. Suppose also that*

(a) *There exist $0 < \delta_1, \delta_2 \in F$ with*

$$\delta_1 a \leqslant \delta_2 \sum_{i+j=n-1} e^i a e^j,$$

(b) *$a \wedge e^n = 0$,*

(c) *$a \wedge e^{n-1} = 0$ if $\alpha_{n-1} > 0$.*

Then there exists $0 < \rho \in F$ and $q(x, y) \in F^+[x, y]$ with $\rho a \ll q(a, e)$.

PROOF. We assume that $n \geq 2$. The coefficient of α^k in $p(\alpha e - a)$ comes from $\alpha_k(\alpha e - a)^k + \alpha_{k+1}(\alpha e - a)^{k+1} + \cdots + \alpha_n(\alpha e - a)^n$ and is

$$\alpha_k e^k + \sum_{m \geq k+1} (-1)^{m-k} \alpha_m \sum_{\substack{i_1 + \cdots + i_t = k \\ j_1 + \cdots + j_t = m-k}} e^{i_1} a^{j_1} \cdots e^{i_t} a^{j_t} = \alpha_k e^k + \mathcal{O}_k(a, e).$$

So

$$0 \leq p(\alpha e - a) = p(-a) + \sum_{k=1}^{n} \left[\alpha_k e^k + \mathcal{O}_k(a, e)\right] \alpha^k$$

$$= p(-a) + [p(\alpha e) - p(0)] + \sum_{k=1}^{n-2} \mathcal{O}_k(a, e) \alpha^k - \alpha_n \alpha^{n-1} \sum_{i+j=n-1} e^i a e^j.$$

Thus,

$$0 \leq \alpha_n \delta_1 \alpha^{n-1} a$$

$$\leq \delta_2 \left[h^+(a) + \sum_{k=1}^{n-2} \left(\alpha_k^+ e^k + \mathcal{O}_k^+(a, e)\right) \alpha^k + \alpha_{n-1}^+ \alpha^{n-1} e^{n-1} + \alpha_n^+ \alpha^n e^n \right],$$

where $h(x) = p(-x)$. So if $\alpha \geq 1$,

$$0 \leq \alpha(\delta_1 \alpha_n a) \leq \delta_2 \left[h^+(a) + \sum_{k=1}^{n-2} \left(\alpha_k^+ e^k + \mathcal{O}_k^+(a, e)\right) + \alpha_{n-1}^+ \alpha e^{n-1} + \alpha_n \alpha^2 e^n \right],$$

and by (b) and (c),

$$0 \leq \alpha \rho a \leq q(a, e) \quad \text{with } \rho = \alpha_n \delta_1.$$

COROLLARY. *The following statements are equivalent for the archimedean l-algebra R over F.*

(a) *R is an f-algebra.*

(b) *For each $a \in \{u^+ v^+ \wedge v^-, v^+ u^+ \wedge v^- : u, v \in R\}$ there exists $p(x) \in F[x]$ and a left f-element $e \geq 0$ such that for all $\alpha \in F^+$*

$$(a \wedge e) \vee (ea - a)^- \vee p(\alpha e - a)^- = 0.$$

LEMMA 2. *Let $f(x, y) \in F[x, y]$ be a polynomial such that $f^-(x, y)$ has a monomial of positive degree in x whose degree in y exceeds the degree of $f^+(x, y)$ in y. Suppose that $a, e \in R^+$ with $a \leq ea$ and $f(a, \alpha e) \geq 0$ for each α in a cofinal subset of F^+. Then there exist $0 < \rho \in F$ and $q(x, y) \in F^+[x, y]$ with $\rho a^n e^t \ll q(a, e)$ for some integers $n \geq 1$ and $t \geq 0$.*

PROOF. Write $f(x, y) = -\rho m(x, y) + h(x, y)$, where $\rho > 0$ and $m(x, y)$ is a monomial whose degree in y exceeds the degree in y of $h^+(x, y)$. Since $a^i \leq e^k a^i$ if $i \geq 1$ and $k \geq 0$, $a^n e^t \leq m(a, e)$ where $n \geq 1$ is the degree of x in $m(x, y)$ and $m(x, y)$ ends in y^t. If $m(x, y)$ has degree s in y, then $f(a, \alpha e) \geq 0$ implies

$$0 \leq \rho \alpha^s a^n e^t \leq \rho m(a, \alpha e) \leq h(a, \alpha e) \leq h^+(a, \alpha e).$$

If $\alpha \geq 1$, then

$$\rho \alpha a^n e^t \leq \alpha^{1-s} h^+(a, \alpha e) \leq h^+(a, e) = q(a, e),$$

since $s >$ degree of y in $h^+(x, y)$.

The equivalence of (a) and (b) in the following theorem is given in [**6**, Corollary 4, p. 206] for the case $p(x) = x^2$. Also, it is shown in Theorem 8 of [**6**] that e is a weak order unit precisely when R satisfies $x^+x^- = 0$. Thus, the equivalence of (a) and (b) follows from Lemma 1.

THEOREM 1. *Let R be an archimedean l-algebra over F and suppose that R has a left superunit e which is an f-element. The following statements are equivalent.*

(a) *R is an f-algebra.*

(b) *R is a PPI l-algebra and satisfies the identity $p(x)^- = 0$ for some $p(x) \in F[x]$.*

(c) *R is a PPI l-algebra and satisfies the identity $f(x, y)^- = 0$, where $f(x, y) = -g(x, y) + p(y) + h(x, y)$ is a right k-nice polynomial with $k \geqslant 2$, and y has higher degree in $g(x, y)$ than in $h^+(x, y)$.*

(d) *R satisfies $f(x^+, x^-)^- = 0$ where $f(x, y)$ is a polynomial satisfying the conditions in* (c).

PROOF. (d) → (a). Let $a \wedge e = 0$. Then if $\alpha \geqslant 0$,

$$0 \leqslant g(a, \alpha e) \leqslant p(\alpha e) + h(a, \alpha e) \leqslant |p|(\alpha e) + h^+(a, \alpha e).$$

Since $g(a, \alpha e) \wedge |p|(\alpha e) = 0$, $g(a, \alpha e) \leqslant h^+(a, \alpha e)$. By Lemma 2 $ae' = 0$ and hence $a^2 = 0$ since $a \leqslant e'a$. By [7, Lemma 10], $a \in T$. But T is an archimedean f-algebra with a superunit and hence is reduced. So e is a weak order unit of R and by the remarks preceding the theorem, R is an f-algebra.

Since the equivalence of (a) and (b) has already been noted and since the implications (a) → (c) and (c) → (d) are trivial, the proof is complete.

In view of the theorem in the introduction one might conjecture that the identity $p(x)^- = 0$ could be localized in Theorem 1, namely, replaced by "for each $u \in R$ there exists $p(x)$ with $p(u) \geqslant 0$". The following example shows that this is not possible. Let $R = \mathbf{Q}(\sqrt{2}) = \mathbf{Q} \oplus \mathbf{Q}\sqrt{2}$ as l-groups. Then for $b \geqslant 0$ or $b \leqslant 0$, $p(b) \geqslant 0$ if $p(x) = x^2$; and if $b = p + q\sqrt{2}$ with $pq < 0$,

$$\text{then } p(b) \geqslant 0 \text{ if } p(x) = \left[(p^2 + 2q^2) - x^2\right]^2.$$

Using a polynomial $f(x, y)$ which satisfies the conditions in (c) it is possible to add the following statement as a third equivalence in the corollary.

For each $a \in \{u^+v^+ \wedge v^-, v^+u^+ \wedge v^-: u, v \in R\}$ there is an f-element $e \geqslant 0$ with

$$(a \wedge e) \vee (ea - a)^- \vee f(a, s^+)^- = 0$$

for each s in the convex l-subalgebra generated by e.

We also note that Diem's example [**2**, p. 72] shows that an archimedean l-domain with squares positive need not be an f-ring.

2. Chain conditions on the algebra. Recall that the unital l-ring R with Jacobson radical J is *local* if R/J is a division ring, and *semiperfect* if R/J is left artinian and idempotents may be lifted from R/J to R. Theorem 2 below is given in [**4**] for the case in which R has squares positive.

LEMMA 3. *Let R be a local l-algebra with radical J. Then R is an f-algebra if and only if the inverse of each positive element is positive.*

PROOF. Assume that $(R^+ \setminus J)^{-1} \subseteq R^+$. Let $a \in R^+$ and put $b = a \vee 2$; and suppose that $x \wedge y = 0$. If $b \notin J$, then $b^{-1} \in R^+$ and

$$0 \leqslant b^{-1}(bx \wedge by) \leqslant b^{-1}bx \wedge b^{-1}by = 0.$$

So $bx \wedge by = 0$. If $b \in J$, then $(b-1)^{-1} \in R^+$ and again $(b-1)x \wedge (b-1)y = 0$. Thus

$$0 \leqslant (b-1)x \wedge y \leqslant (b-1)x \wedge (b-1)y = 0$$

and so $bx \wedge y = [(b-1)x + x] \wedge y = 0$. In either case

$$0 \leqslant ax \wedge y \leqslant bx \wedge by = 0.$$

Similarly, $xa \wedge y = 0$ and R is an f-algebra.

THEOREM 2. *Let R be a unital l-ring such that for each $a \in R$ there is an integer $n \geqslant 1$ with $a^n \geqslant 0$. If R is semiperfect, π-regular, left π-regular or an algebraic algebra over a field, then R is an f-ring.*

PROOF. Since the idempotents of R are all positive, they are central and contained in T. If R is semiperfect, then R/J is a direct sum of division rings and, hence, if $1 = e_1 + \cdots + e_m$ is a lifting of the orthogonal idempotents of R/J, then $R = Re_1 \oplus \cdots \oplus Re_m$ as l-rings. So we may assume that R is local. But if $u \in R^+$ is invertible and $u^{-n} \geqslant 0$, then $u^{-1} = u^{n-1}u^{-n} \geqslant 0$. So R is an f-ring by Lemma 3.

Suppose that R is π-regular. So for each $a \in R$ there is an integer t and $b \in R$ with $a^t = a^t b a^t$; hence $e = ba^t$ is idempotent and $Ra^t = Re$. We may assume that R is a subdirectly irreducible l-ring. But then R is an indecomposable l-ring and hence $e = 0$ or 1. Since $e = 1$ if and only if a is a unit, the nonunits form a nil ideal. In particular, R is local and hence an f-ring by Lemma 3.

If R is left π-regular, that is, each chain $Ra \supseteq Ra^2 \supseteq \cdots$ is finite, and $a \in R^+$, put $b = a \vee 1$. Then for some integer m and $x \in R$, $b^m = xb^{m+1}$. Thus $(1 - xb)b^m = 0$. If $(1 - xb)^n \geqslant 0$, then $(1 - xb)^n b^m = 0$ and hence $(1 - xb)^n = 0$ since $b \geqslant 1$. But then $xb = 1 - (1 - xb)$ is a unit, and therefore so is b. Since $b, b^{-1} \in R^+$, as in the proof of Lemma 3, we see that $a \in T$ and hence R is an f-ring.

Since an algebraic algebra is π-regular the proof is complete.

Let F be a totally ordered field and let F_n be the canonically ordered $n \times n$ triangular matrix l-algebra over F. So

$$F_n = \{(a_{ij}) \colon a_{ij} = 0 \text{ if } i > j\}$$

and

$$F_n^+ = \{(a_{ij}) \in F_n \colon a_{ij} \geqslant 0 \text{ for each } i \text{ and } j\}.$$

It can be shown that the F-l-algebra R is isomorphic to F_2 if and only if R satisfies the following three conditions:

(i) R is noncommutative and 3-dimensional over F.

(ii) $\{a \in R \colon a^m = 0\}$ is a 1-dimensional l-ideal.

(iii) R satisfies the identity $((x^2)^-)^2 = 0$.

Using the identity $((x^2)^-)^n = 0$, what is the analogous characterization of F_n?

References

1. G. Birkhoff and R. S. Pierce, *Lattice-ordered rings*, Ann. Acad. Brasil. Ciênc. **28** (1956), 41–69.

2. J. E. Diem, *A radical for lattice-ordered rings*, Pacific J. Math. **25** (1968), 71–82.

3. H. J. Shyr and T. M. Viswanathan, *On the radicals of lattice-ordered rings*, Pacific J. Math. **54** (1974), 257–260.

4. S. A. Steinberg, *On lattice-ordered rings in which the square of every element is positive*, J. Austral. Math. Soc. Ser. A **22** (1976), 362–370.

5. ______, *Radical theory in lattice-ordered rings*, Sympos. Math. **21** (1971), 379–400.

6. ______, *Identities and nilpotent elements in lattice-ordered rings*, Ring Theory (S. K. Jain, Ed.), Dekker, New York, 1977.

7. ______, *Unital l-prime lattice-ordered rings with polynomial constraints are domains*, Trans. Amer. Math. Soc. **276** (1983), 145–164.

Department of Mathematics, University of Toledo, Toledo, Ohio 43606