PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 89, Number 2, October 1983

ON SPECTRAL SYNTHESIS FOR SETS OF THE FORM $E = \overline{\operatorname{int}(E)}$

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ABSTRACT. The existence of a Helson set disobeying spectral synthesis is combined with the modified Herz criterion to construct a subset E of the circle such that spectral synthesis holds for E and fails for ∂E.

In this note we study the spectral synthesis properties of sets E in the circle group T for which $E = \overline{\operatorname{int}(E)}$. We use the notation of [**2**]. For $E \subseteq T$, let $\operatorname{int}(E)$ denote the set of interior points of E. A closed set E is a set of spectral synthesis, or an S-set, if, for any pseudomeasure S having support in E, there is a net of measures $\{\mu_\alpha\}$ supported by E so that $\mu_\alpha \xrightarrow{w^*} S$. A set E is a Helson set if there exists a number B, the Helson constant of E, so that $\|\mu\| \leqslant B\|\mu\|_{PM}$ for all $\mu \in M(E)$. In the case that E is both a Helson set and an S-set, then every pseudomeasure supported by E is necessarily a measure (see [**2**, p. 92]). We prove the following result.

THEOREM 1. *There is a closed set $E \subseteq T$ that satisfies spectral synthesis and yet spectral synthesis fails for the boundary set ∂E.*

The set E will have the form $E = \overline{\operatorname{int}(E)}$ and will satisfy a modified Herz criterion: there exists $0 < \varepsilon < \frac{1}{2}$ and a sequence of positive integers $\{p_k\}_{k=1}^{\infty}$ tending to infinity so that the sets

$$(1) \qquad H(E, p_k, \varepsilon) = \{x = 2\pi n/p_k \colon n \in Z \text{ and } \operatorname{dist}(x, E) < 2\pi(1-\varepsilon)/p_k\}$$

are all contained in E. This ensures that for every $S \in PM(E)$ there is a sequence $\{\mu_k\}_{k=1}^{\infty}$ of measures supported by E satisfying $\mu_k \xrightarrow{w^*} S$ and $\|\mu_k\|_{PM} \leqslant B\|S\|_{PM}$, where the constant B depends only on the set E and not the particular pseudomeasure [**2**, p. 77]. This result is not new in that the unit ball $E^n = \{x \in \mathbf{R}^n \colon |x| \leqslant 1\}$ satisfies a strong form of spectral synthesis, and it is well known that the boundary set, the unit sphere $S^{n-1} = \{x \in \mathbf{R}^n \colon |x| = 1\}$, is a non-$S$-set for $n > 2$. Our result is new for the group T.

We first give a lemma. We say that a set $F \subseteq T$ is independent if it is independent in $\mathbf{R}$ over the rationals, that is, given integers $n_1, n_2, \ldots, n_m$ and distinct points $x_1, x_2, \ldots, x_m$ in F, then $n_1x_1 + \cdots + n_mx_m = 0$ implies $n_1 = n_2 = \cdots = n_m = 0$. Let $F \subseteq T$ be an independent set that contains no rational multiples of π, and let $\{q_k\}_{k=1}^{\infty}$ be an increasing sequence of positive integers tending to infinity. Given an

Received by the editors January 24, 1983.
1980 *Mathematics Subject Classification*. Primary 43A45; Secondary 43A46.

0002-9939/83 $1.00 + $.25 per page

element $y \in T$, it is easy to find x_1 and x_2 disjoint from $F \cup \{r\pi: r \text{ rational}\}$ so that $F \cup \{x_1, x_2\}$ is independent, y lies in the interval $I = [x_1, x_2]$, and $I \cap F = \varnothing$. Using a well-known method of constructing perfect, independent sets (see [**3**, pp. 101–102]), we can find x_1 and x_2 that satisfy the above conditions and the further condition that some subsequence $\{q_k^1\}_{k=0}^{\infty}$ of $\{q_k\}_{k=1}^{\infty}$ exists for which the sets $H(I, q_k^1, \frac{1}{4})$, $k \geqslant 0$, are contained in I. The same argument allows us to prove

LEMMA 2. *Let F_0 be an independent set in T that contains no rational mutiples of π, let $\{q_k\}_{k=0}^{\infty}$ be an increasing sequence of positive integers tending to infinity, and let H be a finite set with $F_0 \cap H = \varnothing$. Then there exist disjoint intervals $I_1, I_2, \ldots, I_m$ with $I = \bigcup_{j=1}^{m} I_j$ and $\partial I \cap \{r\pi: r \text{ rational}\} = \varnothing$ satisfying*

(i) $I \cap F_0 = \varnothing$,

(ii) $F_0 \cup \partial I$ *is independent*,

(iii) $H \subseteq I \subseteq H + (-2\pi/4q_0, 2\pi/4q_0)$,

(iv) *there exists a subsequence $\{q_k^1\}_{k=0}^{\infty}$ of $\{q_k\}_{k=1}^{\infty}$ so that the sets $H(I, q_k^1, \frac{1}{4})$, $k \geqslant 0$, are contained in I.*

PROOF OF THE THEOREM. Let F be an independent Helson set in T for which spectral synthesis fails [**2**, p. 118]. Since F is independent, we can assume that F contains no rational multiples of π. We use the well-known fact that if F is a given Helson set, then for any finite independent set H, $F \cup H$ is a Helson set whose Helson-set constant is bounded by a fixed constant B depending only on the Helson-set constant of the set F (see [**2**, p. 51]). We define inductively sets I_n, $n \geqslant 1$, each of which is a finite union of closed intervals, and a sequence $\{p_k\}_{k=1}^{\infty}$ of positive integers. Let $p_1 = q_0 = 2$, and use Lemma 2 to obtain I_1 and a subsequence $\{q_k^1\}_{k=0}^{\infty}$ of positive integers. After having chosen sets $I_1, \ldots, I_{n-1}$ and integers $p_1, \ldots, p_{n-1}$ and obtaining a subsequence $\{q_k^{n-1}\}_{k=0}^{\infty}$, choose $p_n \in \{q_k^{n-1}\}$ large enough so that

$$\frac{2\pi}{p_n} < 10 \min\left\{\operatorname{dist}(x, F): x \in \bigcup_{j=1}^{n-1} I_j\right\}. \tag{2}$$

Let H_n denote the set $H(F, p_n, 0)$ and apply the lemma with $q_0 = p_n$ to obtain a finite collection of intervals whose union I_n satisfies the conclusions of Lemma 2 with $F_0 = F \cup \bigcup_{k=1}^{n-1} \partial I_k$, $H = H_n$, and some subsequence $\{q_k^n\}_{k=0}^{\infty}$ of $\{q_k^{n-1}\}_{k=0}^{\infty}$.

Now define E as $E = F \cup \bigcup_{n=1}^{\infty} I_n$. It is clear that $E = \overline{\bigcup I_n} = \overline{\mathrm{int}(E)}$. We claim that E satisfies a modified Herz criterion for the sets (1) for the sequence $\{p_n\}$ and for $\varepsilon = \frac{1}{4}$. Let $\varepsilon = \frac{1}{4}$, $k \in Z$ and $x = 2\pi k/p_n$ satisfy $\operatorname{dist}(x, E) < 2\pi(1-\varepsilon)/p_n$. If $x \in H_n \subseteq I_n \subseteq E$, there is nothing to prove, so assume $x \notin H_n$. Then $\operatorname{dist}(x, F) > 2\pi/p_n$, and so $\operatorname{dist}(x, I_m) < 2\pi(1-\varepsilon)/p_n$ for some m. Since (2) implies that $I_m \subseteq F + (-2\pi/p_n, 2\pi/p_n)$ for $m > n$, and since $m = n$ implies $x \in H_n$, we in fact have $m < n$. Property (iv) and the fact that $p_n \in \{q_k^m\}_{k=1}^{\infty}$ for $m < n$ now forces $x \in I_m \subseteq E$. Thus, E is a set of synthesis.

To finish the proof we show that $\partial E = F \cup \bigcup_{n=1}^{\infty} \partial I_n$ is a Helson non-S-set. Let $\mu \in M(\partial E)$ and $\varepsilon > 0$ be given. Since $\bigcup_{n=1}^{\infty} \partial I_n$ is a countable set, we can find an integer N so that the measure μ_N, the restriction of μ to the set $F \cup \bigcup_{n=1}^{N} \partial I_n$, has

$\|\mu - \mu_N\| < \varepsilon$. By construction, $\bigcup_{n=1}^{N} \partial I_n$ is a finite independent set, and so we obtain

$$\|\mu\| < \|\mu_N\| + \varepsilon < B\|\mu_N\|_{PM} + \varepsilon < B\|\mu\|_{PM} + B\varepsilon + \varepsilon.$$

Since ε and μ are arbitrary, $\|\mu\| \leqslant B\|\mu\|_{PM}$ for all measures μ supported by ∂E, i.e., ∂E is a Helson set. Since there exists an $S \in PM(F) \subseteq PM(\partial E)$ that is not a measure, this proves that ∂E is a non-S-set.

REMARKS. 1. The proof of the theorem is easily adapted so the set E satisfies a modified Herz criterion with sets $H(E, p_k, \varepsilon)$ in (1) for any ε with $0 < \varepsilon < \frac{1}{2}$.

2. A similar proof yields the existence of sets E which are the closures of their interiors and for which spectral synthesis fails. For let F be a non-S-set in T. Then there exists a $\phi \in A(T)$ and an $S \in PM(F)$ satisfying $\phi = 0$ on F and $\langle S, \phi \rangle \neq 0$ [**2**, p. 69]. If $\{x_n\}_{n=1}^{\infty} \subseteq F$ is dense in F, choose $y_n \notin F$, $n \geqslant 1$, with $\operatorname{dist}(x_n, y_n) \to 0$ and $|\phi(y_n)| < 2^{-2n}$. We can now find functions $\phi_n \in A(T)$ with mutually disjoint supports and supports disjoint from F so that $\|\phi_n\| < 2^{-n}$ and $\phi_n = \phi$ on some interval I_n containing y_n. Since the function $\phi - \Sigma\phi_n$ belongs to $A(T)$, vanishes on $E = F \cup \bigcup_{n=1}^{\infty} I_n$, and has $\langle S, \phi - \Sigma\phi_n \rangle = \langle S, \phi \rangle \neq 0$, the set $E = \overline{\operatorname{int}(E)}$ disobeys synthesis. The existence of non-S-sets which are closures of their interiors was originally suggested by Beurling [**1**].

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