PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 89, Number 2, October 1983

THE UNIQUENESS OF MULTIPLICATION IN FUNCTION ALGEBRAS

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ABSTRACT. Let A be a function algebra. We prove that the original multiplication of A is a unique multiplication on the underlying Banach space of A which produces a Banach algebra with the same unit as the original one.

Let A be a complex Banach algebra with unit. We denote by $\mathbf{1}$ the unit of A, the norm by $\|\cdot\|$ and the product of f and g by fg or $f \cdot g$. By the definition of a Banach algebra, for any elements f, g in A we have

$$\|f \cdot g\| \leq \|f\|\|g\| \tag{1}$$

and

$$\mathbf{1} \cdot f = f. \tag{2}$$

Suppose now that A is uniform algebra, that is, A is a commutative Banach algebra with unit and $\|f^2\| = \|f\|^2$ for all f in A. Our goal is to prove that there exists exactly one (associative) multiplication on the Banach space A which satisfies (1) and (2). This result follows upon considering a more general situation.

By an ε-deformation of A we mean an associative multiplication $\times$ on the Banach space A such that

$$\|f \times g - f \cdot g\| \leq \varepsilon\|f\|\|g\| \quad \text{for all } f,\ g \text{ in } A. \tag{3}$$

This definition was formulated by Johnson **[2]** (see also **[3, 4]**). He investigates whether all multiplications on a Banach algebra A near the given multiplication share some of the properties of the original one. Small deformations of function algebras were studied deeply by R. Rochberg **[5]**.

If $\times$ is an ε-deformation of the multiplication of a Banach algebra A then for all f, g in A

$$\|f \times g\| \leq (1 + \varepsilon)\|f\|\|g\| \tag{4}$$

and

$$\|\mathbf{1} \times f - f\| \leq \varepsilon\|f\|. \tag{5}$$

Our main theorem shows that for uniform algebras the converse implication also holds.

Received by the editors October 5, 1982.

1980 *Mathematics Subject Classification*. Primary 46J10; Secondary 46J35.

Key words and phrases. Function algebras, perturbations of multiplication, Nagasawa's Theorem, uniqueness of multiplication.

THEOREM 1. *Suppose $(A, \cdot)$ is a complex uniform algebra. There are positive constants ε_0, c which do not depend on A such that for any $0 \leqslant \varepsilon \leqslant \varepsilon_0$ and any multiplication with unit on A satisfying the conditions*

(i) $\|f \times g\| \leqslant (1 + \varepsilon)\|f\|\|g\|$ *and*

(ii) $\|\mathbf{1} \times f - f\| \leqslant \varepsilon\|f\|$ *for all f, g in A, we have*

(iii) $\|f \times g - f \cdot g\| \leqslant c\sqrt{\varepsilon}\|f\|\|g\|$ *for all f, g in A. Moreover the new multiplication $\times$ is commutative.*

PROOF. If $\varepsilon_0 < 1$ then the condition (ii) implies that the operator T: $A \to A$: $f \mapsto \mathbf{1} \times f$ is an isomorphism so there exists an element e of A such that $\mathbf{1} \times e = \mathbf{1}$. It is easy to check that e is the unit of the algebra $(A, \times)$ and that the element $\mathbf{1}$ is invertible in this algebra. A simple computation using (i) and (ii) proves that

$$\|f \times g - \mathbf{1}^{-1} \times f \times g\| \leqslant \frac{2(1+\varepsilon)^2\varepsilon}{1-\varepsilon}\|f\|\|g\|$$

for all f, g in A. Hence the multiplication $\hat{\times}$ defined by $f \hat{\times} g = \mathbf{1}^{-1} \times f \times g$ has the same unit as the original multiplication of the function algebra A and the multiplication denoted by $\hat{\times}$ is a $k\varepsilon$-deformation of the multiplication $\times$. This proves that without loss of generality we may assume that the element $\mathbf{1}$ is a common unit of both multiplications $\times$ and .

Let us now introduce some notation.

By ∂A and $\operatorname{Ch} A$ we denote the Shilov and the Choquet boundaries of A, respectively. Let

$$\Omega = \left\{x + iy \in \mathbf{C}\colon \left(x - \tfrac{1}{2}\right)^2 + \left(y - \tfrac{1}{2}\right)^2 < \tfrac{1}{2}\right\}$$
$$\cap \left\{x + iy \in \mathbf{C}\colon \left(x - \tfrac{1}{2}\right)^2 + \left(y + \tfrac{1}{2}\right)^2 < \tfrac{1}{2}\right\}$$

and for $r > 0$

$$D(r) = \left\{x + iy \in \mathbf{C}\colon x^2 + y^2 < r^2\right\}.$$

Notice that without loss of generality we may assume that A is an algebra of continuous functions on ∂A.

Fix $\delta > 0$, and let κ: $\overline{D(1)} \to \overline{\Omega}$ be a continuous map of $\overline{D(1)}$ onto $\overline{\Omega}$ such that κ is analytic on $\overline{D(1)}$ and

$$\kappa(1) = 1 \quad \text{and} \quad \kappa(0) = \delta/2.$$

Let $V \subset \mathbf{C}$ be a neighborhood of 0 such that

$$\kappa(V) \subset \overline{\Omega} \cap D(\delta).$$

Now fix any point $s_0 \in \operatorname{Ch} A$ and any of its neighborhoods $U \subset \partial A$, and let $f \in A$ be such that

$$\|f\| = f(s_0) = 1 \quad \text{and} \quad f(\partial A - U) \subset V.$$

The function $\kappa \circ f \in A$ has the following properties:

(a) $\kappa \circ f(\partial A) \subset \overline{\Omega}$;

(b) $\|\kappa \circ f\| = \kappa \circ f(s_0) = 1$;

(c) $\kappa \circ f(\partial A - U) \subset \overline{\Omega} \cap D(\delta)$.

Hence for any $s_0 \in \underline{\mathrm{Ch}} A$ there exists a net $(f_\alpha) \subset A$ such that

(A) $f_\alpha(\partial A) \subset \overline{\Omega}$,

(B) $\|f_\alpha\| = f_\alpha(s_0) = 1$,

(C) (f_α) tends uniformly to zero on the compact subsets of the set $\partial A - \{s_0\}$.

Using the net (f_α) we define

$$g'_\alpha = f_\alpha + i(\mathbf{1} - f_\alpha), \qquad g''_\alpha = f_\alpha - i(\mathbf{1} - f_\alpha).$$

By direct computation

$$g'_\alpha \times g'_\beta = f_\alpha + f_\beta - \mathbf{1} + i(f_\alpha + f_\beta - 2 f_\alpha \times f_\beta).$$

Further observe that, by the definition of Ω, we have

$$\|g'_\alpha\| = \sup_{s \in \partial A} |f_\alpha(s) + i(1 - f_\alpha(s))| \leqslant \sup_{z \in \Omega} |z + i(1 - z)| = 1.$$

Hence from (i) we get

(6) $$1 + \varepsilon \geqslant \|g'_\alpha \times g'_\beta\| \geqslant |g'_\alpha \times g'_\beta(s_0)| = |1 + 2i(1 - f_\alpha \times f_\beta(s_0))|.$$

The same computations for the functions g''_α and g''_β show that

(7) $$1 + \varepsilon \geqslant \|g''_\alpha \times g''_\beta\| \geqslant |g''_\alpha \times g''_\beta(s_0)| = |1 - 2i(1 - f_\alpha \times f_\beta(s_0))|.$$

Inequalities (6) and (7) can be satisfied simultaneously only if

(8) $$|1 - f_\alpha \times f_\beta(s_0)| \leqslant \sqrt{\varepsilon/2 + \varepsilon^2/4} \leqslant \sqrt{\varepsilon}.$$

Now for any $g \in A$ define two functionals $T^l_g: A \to \mathbf{C}$ and $T^r_g: A \to \mathbf{C}$ by

$$T^l_g(f) = g \times f(s_0), \qquad T^r_g(f) = f \times g(s_0).$$

For each $g \in A$ fix two regular measures μ^l_g and μ^r_g on ∂A such that

$$\mu^l_g(f) = T^l_g(f), \qquad \operatorname{var}(\mu^l_g) = \|T^l_g\|,$$
$$\mu^r_g(f) = T^r_g(f), \quad \operatorname{var}(\mu^r_g) = \|T^r_g\| \quad \text{for all } f \text{ in } A.$$

Inequality (8) shows that

(9) $$|\mu^l_{f_\alpha}(f_\beta) - 1| \leqslant \sqrt{\varepsilon} \quad \text{for any } \alpha \text{ and all } \beta.$$

By the definition of (f_α) we get

$$|\mu^l_{f_\alpha}(\{s_0\}) - 1| \leqslant \sqrt{\varepsilon}.$$

Hence, because $\operatorname{var}(\mu^l_{f_\alpha}) = \|T^l_{f_\alpha}\| = 1 + \varepsilon$, the measure $\mu^l_{f_\alpha}$ is of the form

(10) $$\mu^l_{f_\alpha} = \delta_{s_0} + \Delta\mu^l_{f_\alpha}$$

where δ_{s_0} is a Dirac measure concentrated at the point s_0 and $\operatorname{var}(\Delta\mu^l_{f_\alpha}) \leqslant 3\sqrt{\varepsilon}$.

Now let g_0 be any element of A such that $\|g_0\| = 1 = g_0(s_0)$. By (10) we get

$$\mu^r_{g_0}(f_\alpha) = f_\alpha \times g_0(s_0) = \mu^l_{f_\alpha}(g_0)$$
$$= g_0(s_0) + \Delta\mu^l_{f_\alpha}(g_0) = 1 + \Delta\mu^l_{f_\alpha}(g_0).$$

Hence

$$|\mu^r_g(f_\alpha) - 1| \leqslant 3\sqrt{\varepsilon}.$$

In the same way as previously, we get

$$\mu^r_{g_0} = \delta_{s_0} + \Delta\mu^r_{g_0} \quad \text{where } \operatorname{var}\left(\Delta\mu^r_{g_0}\right) \leqslant 7\sqrt{\varepsilon}\,. \tag{11}$$

Using this we can estimate the norm of $g_0 \times g_0$ from below.

$$\begin{aligned}\|g_0 \times g_0\| &\geqslant |g_0 \times g_0(s_0)| = |\mu^r_{g_0}(g_0)| = |1 + \Delta\mu^r_{g_0}(g_0)|\\ &\geqslant 1 - 7\sqrt{\varepsilon}\,.\end{aligned}$$

Because s_0 is an arbitrary point of Ch A this proves that

$$\|g \times g\| \geqslant \left(1 - 7\sqrt{\varepsilon}\right)\|g\|^2 \quad \text{for any } g \text{ in } A. \tag{12}$$

As an immediate consequence of (12) we conclude that the spectral radius of any element g of the algebra $(A, \times)$ is not less than $(1 - 7\sqrt{\varepsilon})\|g\|$. Hence by a theorem of Hirschfeld and Żelazko [**1**] one obtains the commutativity of the multiplication $\times$ if $1 - 7\sqrt{\varepsilon} > 0$.

Applying (12) for $g = f_\alpha$ and using the commutativity of $\times$ we get that there exists a linear and $\times$-multiplicative functional F_α such that $|F_\alpha(f_\alpha)| \geqslant 1 - 7\sqrt{\varepsilon}$. For any f in A of norm equal one we have

$$(1 + \varepsilon)\|F_\alpha\| \geqslant \|F_\alpha\|\|f \times f\| \geqslant |F_\alpha(f \times f)| = |F_\alpha(f)|^2,$$

hence

$$(1 + \varepsilon)\|F_\alpha\| \geqslant \|F_\alpha\|^2, \quad \text{so } \|F_\alpha\| \leqslant 1 + \varepsilon.$$

Let ν_α be a regular measure on ∂A which represents the functional F_α and such that $\operatorname{var}(\nu_\alpha) = \|F_\alpha\|$. We have

$$\begin{cases} |\nu_\alpha(f_\alpha)| \geqslant 1 - 7\sqrt{\varepsilon}\,,\\ \operatorname{var}(\nu_\alpha) \leqslant 1 + \varepsilon,\\ \nu_\alpha(\mathbf{1}) = 1 \quad \text{for all indices } \alpha. \end{cases} \tag{13}$$

Taking a net finer than (f_α) and using the weak $*$ compactness of ∂A we can assume, without loss of generality, that the net (ν_α) is weak $*$ convergent to the measure ν_0. The measure ν_0 also represents a linear and $\times$-multiplicative functional F_0 on A. From (13) we derive that the measure ν_0 is of the form

$$\nu_0 = \delta_{s_0} + \Delta\nu_{s_0} \quad \text{where } \operatorname{var}\left(\Delta\nu_{s_0}\right) \leqslant c_1\sqrt{\varepsilon}\,. \tag{14}$$

From (14) for any f, g in A we find

$$\begin{aligned} f \times g(s_0) + \Delta\nu_{s_0}(f \times g) &= \nu_0(f \times g) = \nu_0(f) \cdot \nu_0(g)\\ &= \left(f(s_0) + \Delta\nu_{s_0}(f)\right) \cdot \left(g(s_0) + \Delta\nu_{s_0}(g)\right)\\ &= f(s_0) \cdot g(s_0) + \Delta\nu_{s_0}(f) \cdot g(s_0) + f(s_0) \cdot \Delta\nu_{s_0}(g) + \Delta\nu_{s_0}(f) \cdot \Delta\nu_{s_0}(g_0). \end{aligned}$$

Hence

$$\begin{aligned} |f \times g(s_0) - f \cdot g(s_0)| &\leqslant \operatorname{var}(\Delta\nu_{s_0})\left(2\|f\|\|g\| + \operatorname{var}(\Delta\nu_{s_0}) \cdot \|f\|\|g\|\right)\\ &\leqslant c_1\sqrt{\varepsilon}\left(2 + c_1\sqrt{\varepsilon}\right) \cdot \|f\|\|g\| = c\sqrt{\varepsilon}\|f\|\|g\|. \end{aligned}$$

Because s_0 is an arbitrary point in Ch A, which is a dense subset of ∂A, the above statement proves (iii) and ends the proof of the theorem.

COROLLARY 1. *Suppose* $(A, \cdot)$ *is a complex function algebra. Let* $\times$ *be any associative multiplication on the Banach space* A *with the same unit and such that* $(A, \times)$ *is a Banach algebra (this means such that* $\|f \times g\| \leq \|f\|\|g\|$ *for all* f, g *in* A). *Then the new multiplication* $\times$ *and the original one coincide.*

The above corollary can be also formulated in the following way, giving a generalization of Nagasawa's Theorem.

COROLLARY 2. *Let* $(A, \cdot)$ *be a complex function algebra with unit* $\mathbf{1}_A$ *and let* B *be any Banach algebra with unit* $\mathbf{1}_B$. *Suppose* T *is a linear isometry from* A *onto* B *such that* $T\mathbf{1}_A = \mathbf{1}_B$. *Then* T *is an algebra isomorphism of* A *and* B.

Notice that we have only considered complex Banach algebras. The theorem and the corollaries are not valid for real function algebras, even in two dimensions. To prove this let $A = (\mathbf{R}^2, \cdot, \|\cdot\|_\infty)$ be the two dimensional real function algebra and let $\rho_t\colon \mathbf{R}^2 \times \mathbf{R}^2 \to \mathbf{R}^2$.

$$\rho_t((x, y), (x', y')) = (xx' - t(x-y)(x'-y'), yy' - t(x-y)(x'-y')).$$

A direct computation shows that for any $0 \leq t \leq 1/2$ the bilinear map ρ_t is a commutative, associative multiplication on $\mathbf{R}^2$ such that $\|\rho_t\| = 1$ and

$$\rho_t((1,1),(x, y)) = (x, y) \quad \text{for any } (x, y) \in \mathbf{R}^2.$$

Let us end the paper with a natural problem arising from Corollary 1.

PROBLEM. Characterize those Banach spaces A and elements e which admit a unique multiplication $\times$ on A so that $(A, \times)$ is a Banach algebra with unit e.

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