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ON THE EXISTENCE AND BOUNDARY BEHAVIOR OF SOLUTIONS TO A CLASS OF NONLINEAR DIRICHLET PROBLEMS

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ABSTRACT. In this paper, we first extend the well-known method of super- and sub-solutions for elliptic boundary value problems to L^∞-boundary functions. Then we apply this method to investigate the solvability and the boundary behavior of solutions to some nonlinear elliptic equations, some Fatou-type results are obtained.

Let $L = \sum_{i,j=1}^n \partial_{x_i}(a_{ij}(x)\partial_{x_j})$, where $a_{ij}(x) = a_{ji}(x) \in C^\infty(\mathbf{R}^n)$, and $\sum_{i,j} a_{ij}(x)\xi_i\xi_j \geqslant \lambda|\xi|^2$. Let Ω be a bounded, smooth domain in $\mathbf{R}^n$, and $f(x, u)$ either Lipschitz in u (and C^α in both variables) or increasing in u (and C^α in both variables), which satisfies $f(x, u) = O(|u|^p)$ at $u = 0$, for some $p > 1$, uniformly in x. Our main interest in this note is to study the Dirichlet problem for the operator $Lu + f(x, u)$ in Ω, with boundary data $g \in L^\infty(\partial\Omega)$. We accomplish this by extending to our setting the classical method of super- and sub-solutions. This method goes back to Bieberbach (see the last paragraph of [**K, W**] for further historical comments). More recently, this method was used in [**S**], in a manner very similar to ours.

Finally, the results on the boundary behavior of solutions that we obtain are identical to (and follow from) those in the linear theory, as in [**W** or **J, K**].

THEOREM 1. *There exists a number $\varepsilon_0 > 0$ such that, if $\|g\|_{L^\infty(\partial\Omega)} \leqslant \varepsilon_0$, then there exists a function u in $C^{2,\alpha}(\Omega)$, which satisfies $Lu + f(x, u) = 0$ in Ω, and such that for a.e. $Q(d\sigma)$ on $\partial\Omega$, $u(x)$ converges to $g(Q)$ as X converges to Q nontangentially (i.e., $\lim_{x\in\Gamma_\alpha(Q)x\to Q} u(x) = g(Q)$ for every $\alpha > 0$, where $\Gamma_\alpha(Q) = \{x \in \Omega: |x - Q| \leqslant (1 + \alpha)\operatorname{dist}(x, \partial\Omega)\}$, for a.e. $Q(d\sigma) \in \partial\Omega$).*

PROOF. We first note that since $\bar\Omega$ is compact, we can find $\varphi \in C^\infty(\bar\Omega)$ such that $\varphi(x) > 0$ for every $x \in \bar\Omega$, and such that $L\varphi(x) < 0$ for every $x \in \bar\Omega$. (Simply solve $L\psi = -1$ in Ω, $\psi|_{\partial\Omega} \equiv 0$. By the minimum principle, $\psi \geqslant 0$ in Ω, therefore, $\varphi = \psi + 1$ satisfies all the required properties.) Consider now $\lambda\varphi$, where $\lambda > 0$. Then,

$$L(\lambda\varphi) + f(x, \lambda\varphi) = \lambda L(\varphi) + O((\lambda\varphi)^p) = \lambda(L\varphi + \lambda^{(p-1)}O(\varphi^p)) < 0$$

if $\lambda \leqslant \lambda_0$, and λ_0 is small enough. Fix such a λ_0 and let $\mu = \lambda_0\varphi$. Then, clearly $\mu \in C^\infty(\bar\Omega)$, $\mu(x) > 0$ for every $x \in \bar\Omega$, and $L\mu + f(x, \mu) < 0$ for $x \in \bar\Omega$. Define now

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$\varepsilon_0 = \min_{x \in \bar{\Omega}} \mu(x)$. We will first show that if $0 < \varepsilon \leqslant \varepsilon_0$, then we can find $u \in C^{2,\alpha}(\bar{\Omega})$, so that $Lu + f(x, u) = 0$ in Ω, $u(x) > 0$ in Ω and $u|_{\partial\Omega} = \varepsilon$. We first note that, without loss of generality, we can assume that $f(x, u)$ is increasing in u. (Substitute f by $f + Mu$, and L by $L - Mu$, where M is large.) We will assume f to be increasing in the rest of the proof.

Now let v_1 be the solution to the linear problem

$$Lv_1 + f(x, \mu) = 0, \qquad v_1|_{\partial\Omega} = \varepsilon.$$

Then, as f is C^α, $v_1 \in C^{2,\alpha}$, moreover, as $\mu \geqslant 0$, and hence $f(x, \mu) \geqslant 0$, the minimum principle shows that $v_1 \geqslant \varepsilon$ in Ω. In addition, $L(v_1 - \mu) = -L\mu - f(x, \mu) > 0$ and $(v_1 - \mu)|_{\partial\Omega} \leqslant 0$. Hence, $v_1 \leqslant \mu$. Inductively, define v_{k+1} as the solution of

$$Lv_{k+1} + f(x, v_k) = 0, \qquad v_{k+1}|_{\partial\Omega} = \varepsilon.$$

Then, arguing as above we can check that $\varepsilon \leqslant v_{k+1} \leqslant v_k \leqslant \cdots \leqslant \mu$. Now, set $u(x) = \lim_{k \to \infty} v_k(x)$. We first show that $Lu = -f(x, u)$ in Ω in the sense of distributions. In fact, if $\varphi \in C_0^\infty(\Omega)$, then,

$$\int_\Omega (Lu + f(x, u))\varphi = \int_\Omega (uL\varphi + f(x, u)\varphi) = \lim_{k \to \infty} \int_\Omega (v_k L\varphi + f(x, v_k)\varphi) = 0,$$

by dominated convergence. Moreover, it is easy to see that the v_k are uniformly bounded in $W^{2,p}(\Omega)$ for every p, $1 \leqslant p < \infty$. Therefore, $u \in W^{2,p}(\Omega)$, for every p, $1 \leqslant p < \infty$, and hence $Lu = -f(x, u)$ in the $W^{2,p}$ sense. Since $f \in C^\alpha$ it is easy to conclude now that $u \in C^{2,\alpha}(\bar{\Omega})$ is a classical solution of $Lu + f(x, u) = 0$, $u|_{\partial\Omega} = \varepsilon$.

A similar argument shows the existence of a number $\varepsilon_0 > 0$ so that if $-\varepsilon_0 \leqslant -\varepsilon < 0$, we can find a solution $u \in C^{2,\alpha}(\bar{\Omega})$ to $Lu + f(x, u) = 0$, $u|_{\partial\Omega} = -\varepsilon$. Also we have $u < -\varepsilon$ in Ω.

We are now ready to prove our theorem. Let ε_0 be as above, and $\bar{v}$, $\underline{v} \in C^{2,\alpha}(\bar{\Omega})$ the solutions of $L\bar{v} + f(x, \bar{v}) = 0$ (resp. $L\underline{v} + f(x, \underline{v}) = 0$) and $\bar{v}|_{\partial\Omega} = \varepsilon_0$ (resp. $\underline{v}|_{\partial\Omega} = -\varepsilon_0$), with $\bar{v} \geqslant \varepsilon_0$, $\underline{v} \leqslant -\varepsilon_0$. As before, let v_1 be a bounded solution of

$$Lv_1 + f(x, \bar{v}) = 0, \qquad v_1|_{\partial\Omega} = g,$$

in the sense that $v_1 \in C^{2,\alpha}(\Omega)$, and the boundary values are taken nontangentially a.e. Then the maximum and minimum principle still apply (see for example [**W**]), and we have $\underline{v} \leqslant v_1 \leqslant \bar{v}$, in Ω. As before, we inductively define v_{k+1} as the bounded solution of the linear problem

$$Lv_{k+1} + f(x, v_k) = 0, \qquad v_{k+1}|_{\partial\Omega} = g,$$

where the boundary values are taken nontangentially a.e. By the same argument as before, $\underline{v} \leqslant v_{k+1} \leqslant v_k \leqslant \cdots \leqslant \bar{v}$. Again, set $u(x) = \lim_{k \to \infty} v_k(x)$. Arguing as before, $Lu = -f(x, u)$ in the distribution sense, and also in $W^{2,p}_{\text{loc}}(\Omega)$, and hence $u \in C^{2,\alpha}(\Omega)$ and $Lu + f(x, u) = 0$ in Ω in the classical sense. To see that the boundary values are taken in the desired sense, we argue as follows. Let v be the bounded solution of the linear problem

$$Lv = 0 \quad \text{in } \Omega, \qquad v|_{\partial\Omega} = g,$$

where the boundary values are taken nontangentially a.e. Let $Z \subset \partial\Omega$ be the set of $Q \in \partial\Omega$ such that the nontangential limit for v at Q does not equal $g(Q)$. By [**W**], Z

has surface measure 0. Let $\varepsilon > 0$ be given now, let $Q \in \partial\Omega \setminus Z$, and fix a nontangential approach region $\Gamma_\alpha(Q)$. Then, we claim that there exists $\delta > 0$, depending only on ε, α, Ω, but not on k so that

$$\sup_{x \in \Gamma_\alpha(Q) \cap B_\delta(Q)} |v_k(x) - g(Q)| < \varepsilon,$$

where $B_\delta(Q)$ is the ball of radius δ in $\mathbf{R}^n$, centered at Q. From this claim it is immediate that, for $Q \in \partial\Omega \setminus Z$, $u(x) \to g(Q)$ as x converges nontangentially to Q. To establish the claim, rewrite $v_{k+1} = v + w_k$, where w_k is the Green potential for L of $f(x, v_k)$, i.e. $w_k \equiv 0$ on $\partial\Omega$, and $Lw_k = -f(x, v_k)$. Since $\|f(x, v_k)\|_\infty \leqslant M$, where M is independent of k, standard elliptic estimates for w_k show that $\|w_k\|_{W^{2,p}(\Omega)} \leqslant \tilde{M}$, where $\tilde{M}$ is independent of k, and therefore, $\|w_k\|_{C^\beta(\bar{\Omega})} \leqslant N$, N independent of k, for any β, $0 < \beta < 1$. Therefore, as $w_k|_{\partial\Omega} \equiv 0$, given $\varepsilon > 0$, we can choose $\delta > 0$, independent of k so that

$$\sup_{\bigcup_{Q \in \partial\Omega} B_\delta(Q) \cap \Omega} |w_k(x)| \leqslant c\delta^\beta \leqslant \frac{\varepsilon}{2}.$$

Choosing δ also so that

$$\sup_{x \in \Gamma_\alpha(Q) \cap B_\delta(Q)} |v(x) - g(Q)| \leqslant \frac{\varepsilon}{2}$$

(which is possible by our choice of Z), our claim follows.

REMARKS. (1) If $g \geqslant 0$, and $g > 0$ on a set of positive surface measure of $\partial\Omega$, our construction produces a positive solution in Ω.

(2) If $g \in C(\partial\Omega)$, an easy modification of our argument shows that $u \in C(\bar{\Omega})$, and $u|_{\partial\Omega} = g$ at every point. Likewise, if $g \in C^\beta(\partial\Omega)$, $0 < \beta < 1$, $u \in C^\beta(\bar{\Omega})$.

(3) Our proof also shows that if $Z \subset \partial\Omega$, is a set of 0 surface area for which there exists a bounded function v which is a solution of $Lv = 0$, and which fails to have nontangential limits at every point Q of Z, then we can construct a nonnegative bounded solution u of $Lu + f(x, u) = 0$, which has the same property.

(4) Arguing as in the proof of the last claim in the proof of Theorem 1, using the results of **[W]**, it is possible to show that if u is any solution of $Lu + f(x, u) = 0$, which is bounded in Ω, then u has nontangential limits a.e. on $\partial\Omega$.

(5) Using the results in **[J, K]**, and the estimates for Green potentials in **[M]**, it is possible to extend, modifying the proof only slightly, Theorem 1 and Remarks (1), (2), (4) and (5) after it to the case when Ω is a bounded Lipschitz domain in $\mathbf{R}^n$. Remark (3) also holds in this case, provided that in the case $g \in C^\beta(\partial\Omega)$, we restrict ourselves to $\beta \leqslant \beta_0$, where $\beta_0 > 0$ is a number which depends only on the Lipschitz character of $\partial\Omega$. Also, the smoothness assumptions on the coefficients $a_{ij}(x)$ can be considerably relaxed (for example, it is enough to assume $a_{ij}(x) \in C^1(\mathbf{R}^n)$).

(6) In the proof of Theorem 1, we have actually proved the following extension of the classical super- and sub-solutions method, which seems to be of some independent interest:

Consider the following Dirichlet problem

(D) $$Lu + f(x, u) = 0 \quad \text{in } \Omega, \qquad u|_{\partial\Omega} = g,$$

where Ω is a bounded smooth domain in $\mathbf{R}^n$, $g \in L^\infty(\partial\Omega)$ and f is either Lipschitz in u (and Hölder C^α in both variables) or increasing in u and C^α in (x, u). We say v is a *super-(sub-) solution* of (D) if

$$Lv + f(x, v) \leqslant (\geqslant) 0, \qquad v|_{\partial\Omega} \geqslant (\leqslant) g.$$

Now, we have

PROPOSITION. *If $\varphi \in C^2(\bar{\Omega})$ is a super-solution of* (D) *and $\psi \in C^2(\bar{\Omega})$ is a sub-solution of* (D) *with $\varphi \geqslant \psi$, then* (D) *possesses a solution u with $\varphi \geqslant u \geqslant \psi$.*

The following theorem explains the role of ε_0 in Theorem 1.

THEOREM 2. *Consider the Dirichlet problem*

$$(*) \qquad \Delta u + |u|^p = 0 \quad \text{in } B, \qquad u|_{\partial B} = g,$$

where B is the unit ball in $\mathbf{R}^n$, $p > 1$, $g \in L^\infty(\partial B)$, $g \geqslant 0$ and $g \not\equiv 0$. Then, there exists a number $\varepsilon_0 > 0$ such that if $\|g\|_\infty \leqslant \varepsilon_0$, then $(*)$ has a solution, and such that if

$$\frac{1}{\sigma(\partial B)} \int_{\partial B} g \, d\sigma > \varepsilon_0,$$

then $(*)$ has no solution.

PROOF. First consider the case $g \equiv c > 0$, a constant. It is easy to see (from [G, N, N]) that a solution of $(*)$ must be radially symmetric. It is then not hard to show that there exists an $\varepsilon^* > 0$ such that $(*)$ does not have a solution if $c > \varepsilon^*$. (For $n > 2$, see, for example, p. 260, Theorem 1(ii) in [J, L]. For the case $n = 2$, it may be treated as follows: first find a positive solution u of $(*)$ with $g \equiv 0$, then consider $v(r) = \lambda^{2/(p-1)} u(\lambda r)$ for $0 < \lambda \leqslant 1$, $r \leqslant 1$. This is a one-parameter family of solutions of $(*)$ with $g_\lambda \equiv \lambda^{2/(p-1)} u(\lambda)$. As $\lambda \to 0$, $g_\lambda \to 0$; also, as $\lambda \to 1$, $g_\lambda \to 0$. Thus, we may let $\varepsilon^* = \max_{\lambda \in (0,1)} g_\lambda$.)

Since it is easy to show $(*)$ has a positive solution for *some* $g \equiv c > 0$ (using the same rescaling idea as above), the argument in Theorem 1 shows there exists an $\varepsilon_0 > 0$ such that $(*)$ possesses a solution if $g \equiv c \leqslant \varepsilon_0$ but not otherwise.

The proof of Theorem 1 furnishes existence when $\|g\|_\infty \leqslant \varepsilon_0$. For nonexistence when $1/\sigma(\partial B) \int_{\partial B} g \, d\sigma > \varepsilon_0$, we argue by contradiction. If $(*)$ had a solution u in this case, we set $\bar{u}(r) =$ average of u on the sphere of radius r. Then, by a standard argument (see for example [N]), $\Delta\bar{u} + \bar{u}^p \leqslant 0$, $\bar{u}|_{\partial B} = 1/\sigma(\partial B) \int_{\partial B} g > \varepsilon_0$, which, arguing as in Theorem 1, contradicts the results above.

REFERENCES

[G, N, N] B. Gidas, W.-M. Ni and L. Nirenberg, *Symmetry and related properties via the maximum principle*, Comm. Math. Phys. **68** (1979), 209–243.

[J, K] D. S. Jerison and C. E. Kenig, *The Dirichlet problem in non-smooth domains*, Ann. of Math. (2) **113** (1981), 367–382.

[J, L] D. D. Joseph and T. S. Lundgren, *Quasilinear Dirichlet problems driven by positive sources*, Arch. Rational Mech. Anal. **49** (1973), 241–269.

[K, W] J. L. Kazdan and F. W. Warner, *Curvature functions for open 2-manifolds*, Ann. of Math. (2) **99** (1974), 203–219.

[**M**] C. B. Morrey, *Second order elliptic equations in several variables and Hölder continuity*, Math. Z. **72** (1959), 146–164.

[**N**] W.-M. Ni, *On the elliptic equation* $\Delta u + K(x)u^{(n+2)/(n-2)} = 0$, *its generalizations and applications to geometry*, Indiana Univ. Math. J. **31** (1982), 493–529.

[**S**] D. Sattinger, *Topics in stability and bifurcation theory*, Lecture Notes in Math., vol. 309, Springer-Verlag, Berlin and New York, 1973.

[**W**] K. O. Widman, *On the boundary behavior of solutions to a class of elliptic partial differential equations*, Ark. Mat. **6** (1966), 485–533.

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