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SOME RESULTS CONNECTED WITH A PROBLEM OF ERDÖS. III

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ABSTRACT. It is shown that if E is a subset with more than two points of the real line, then there exists a subset S of the unit interval, such that S has outer Lebesgue measure one and S is of the second Baire category and such that S does not contain a subset similar (in the sense of elementary geometry) to E. This result is related to a conjecture of P. Erdös.

1. Introduction. It is known that if E is a finite subset of $\mathbf{R}$ and A a subset of positive Lebesgue measure, then A contains a subset similar (in the sense of elementary geometry) to E.

P. Erdös [**1**] presented the following conjecture at the problem session of the Fifth Balkan Mathematical Congress (Belgrade, June 24–30, 1974):

CONJECTURE. Let E be an infinite set of real numbers. Then there exists a set of real numbers S of positive Lebesgue measure which does not contain a set E' similar to E.

H. I. Miller [**3**] proved, using the continuum hypothesis, that if E is an uncountable subset of the real line, then there exist subsets S_1 and S_2 of the unit interval, such that S_1 has outer Lebesgue measure one and S_2 is of the second Baire category and such that neither S_1 nor S_2 contains a subset similar to E.

Here we prove without the continuum hypothesis the following theorem.

THEOREM. *Let E be a subset of* $\mathbf{R}$ *with more than two points. There exists a subset S of the unit interval such that*

1. *S has outer Lebesgue measure one.*
2. *S is of the second Baire category at each of its points.*
3. *S does not contain a subset similar to E.*

2. Proof. We may assume that E has three points.

Let Ω be the first ordinal number of the power of the continuum. Suppose $\langle x_\alpha: \alpha < \Omega\rangle$ (resp. $\langle F_\alpha: \alpha < \Omega\rangle$) is a transfinite sequence whose elements are the points of $[0, 1]$ (resp. the perfect subsets of $[0, 1]$). We shall construct by transfinite induction another sequence $\langle y_\alpha: \alpha < \Omega\rangle$, with real numbers, in the following way.

Suppose that if $\beta < \alpha$ we have chosen y_β. Let A_α be the set of $x \in \mathbf{R}$ such that there exists a similarity transformation u with $u(E) \subset \{x\} \cup \{y_\beta: \beta < \alpha\}$. Since the

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cardinality of the set A_α is less than the cardinality of the set $6 \times \{y_\beta: \beta < \alpha\} \times \{y_\beta: \beta < \alpha\}$ we have $|A_\alpha| < 2^{\aleph_0}$. We call y_α the first point in $\langle x_\gamma: \gamma < \Omega \rangle$ contained in $F_\alpha \setminus (A_\alpha \cup \{y_\beta: \beta < \alpha\})$.

Set $S = \{y_\alpha: \alpha < \Omega\}$. It is clear that $S \subset [0, 1]$.

Since $y_\alpha \in S \cap F_\alpha$, S intersects every perfect subset of $[0, 1]$. It is clear that the outer Lebesgue measure of S is one.

S is of second Baire category at each of its points because every uncountable Borel subset of **R** contains a perfect subset [**2**, p. 427].

Finally, by construction, S does not contain a subset similar to E.

References

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