A NOTE ON $\alpha$-COMPACT SPACES

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Abstract. For an infinite cardinal $\alpha$, $m(\alpha)$ denotes the least measurable cardinal, if one exists, not less than $\alpha$. We give easy proofs of generalizations of some results on realcompact spaces. Among these we prove the following generalization of a theorem of A. Kato.

Let $\{X_i: i \in I\}$ be a collection of spaces each having at least two elements. Then the $k$-box product $\prod_{i \in I} X_i$ is $\alpha$-compact if and only if either $X_i$ is $\alpha$-compact for each $i \in I$ and $k \leq m(\alpha)$ or $|I| < m(\alpha)$.

All spaces discussed in this paper are completely regular Hausdorff spaces. Let $X$ be a space and $\alpha$ an uncountable cardinal. Then $Z(X)$ denotes the set of all zero sets in $X$ and $\beta X$ the Stone-Čech compactification of $X$. A family $\mathcal{A}$ of sets has the $\alpha$-intersections property (\alpha-i.p. for short) if $\bigcap \mathcal{B} \neq \emptyset$ whenever $\mathcal{B} \subseteq \alpha$ and $|\mathcal{B}| < \alpha$. We write $\beta_\alpha(X) = \{p \in \beta X: p$ has the \alpha-i.p.$\}$. The space $X$ is said to be $\alpha$-compact if $X = \beta_\alpha X$. See [1 and 4] for a discussion of these spaces and for references to other papers on the subject.

Let $k$ be an uncountable cardinal and let $(X, \tau)$ be a space. $(X, \tau(k))$ denotes the space with basis the family of all intersections of less than $k$ members of $\tau$. Let $\{X_i: i \in I\}$ be a family of spaces. Then $(\prod_{i \in I} X_i)_k$ denotes the $k$-box product of the family. This space has as basis sets of the form $\prod U_i$ where $U_i \neq X_i$ for less than $k$ many $i$ and $U_i$ is open in $X_i$.

Let $\alpha$ be an uncountable cardinal number. Then $m(\alpha)$ stands for the least measurable cardinal such that $\alpha \leq m(\alpha)$ (if one exists). We say that $m$ is measurable if there is a discrete space $A$ with $|A| = m$ and $\beta_m(A) - A \neq \emptyset$.

Theorem 1. Let $p \in \beta_\alpha(X) - X$ and $\mathcal{B} \subseteq p$ such that

(i) $\bigcap \mathcal{B} = \emptyset$, and

(ii) $\mathcal{B}' \subseteq \mathcal{B}$ implies $\bigcap \mathcal{B}' \in Z(X)$.

Then $|\mathcal{B}| \geq m(\alpha)$.

Proof. We may assume that $\mathcal{B}$ has the minimum cardinality with respect to having empty intersection. Let $|\mathcal{B}| = m$. Then $\mathcal{B}$ has the $m$-i.p. In fact if $\mathcal{B}' \subseteq \mathcal{B}$ and $|\mathcal{B}'| < \alpha$ then $\bigcap \mathcal{B}' \in p$. If this were not the case then there would exist $Z \in p$ such that $Z \cap (\bigcap \mathcal{B}') = \emptyset$. Hence the family $\mathcal{B}' \cup \{Z\}$ would be a subset of $p$ with empty intersection, contrary to the minimality of $|\mathcal{B}|$.

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Let $\mathfrak{B} = \{B_a;\ a \in A\}$ with $|\mathfrak{B}| = |A|$. For $D \subseteq A$ let $Z_D = \bigcap_{a \in D} B_a$ and let $q = \{D \subseteq A;\ Z_D \notin p\}$. Then

(i) $Z_{\emptyset} = X \in p \Rightarrow \emptyset \notin q;\ Z_A = \emptyset \notin p \Rightarrow A \in q$.

(ii) $D \notin q \Rightarrow Z_D \notin p$. Then $Z_D \cap Z_{A - D} = Z_{D \cup (A - D)} = Z_A \notin p$. Hence $Z_{A - D} \notin p$. which implies that $A - D \in q$.

(iii) $D \subseteq E$ and $D \in q \Rightarrow Z_D \notin p$ and since $Z_E \subseteq Z_D$ we have $Z_E \notin p$. Hence $E \in q$. From (i), (ii) and (iii) we conclude that $q$ is an ultrafilter on $A$. Since $B_a \in p$ for any $a$ we see that $\{a\} \notin q$. Hence $q$ is free.

(iv) Let $(D_i;\ i \in I) \subseteq q$ where $|I| \leq m$. Then $D_i \in q \Rightarrow A - D_i \notin q \Rightarrow Z_{A - D_i} \notin p$ for all $i \in I \Rightarrow \bigcap Z_{A - D_i} \in p \Rightarrow Z_{\bigcup(A - D_i)} \in p \Rightarrow \bigcup(A - D_i) \in q \Rightarrow \bigcap D_i \notin q$.

Thus $q$ has the $m$-i.p. Hence $m$ is measurable and $m \geq \alpha$. Hence $m \geq m(\alpha)$ which implies that $|\mathfrak{B}| \geq m(\alpha)$ as claimed.

**Corollary 1.** A discrete space $A$ is $\alpha$-compact if and only if $|A| \leq m(\alpha)$.

**Proof.** If $|A| \geq m(\alpha)$ then clearly $\beta_m(A) = A \neq \emptyset$ and so $A$ is not $\alpha$-compact. Conversely, if $p \in \beta_m(A) - A$ then with $\beta=p$ in Theorem 1 we see that $|p| \geq m(\alpha)$. Hence $2^{\omega_1} \geq m(\alpha)$ and since $m(\alpha)$ is strongly inaccessible we have $|A| \geq m(\alpha)$.

**Corollary 2.** A metric space $M$ is $\alpha$-compact if and only if $|M| \leq m(\alpha)$.

**Proof.** Let $M$ be $\alpha$-compact. Then every closed discrete subset (being $\alpha$-compact) has cardinality less than $m(\alpha)$ by Corollary 1. It is well known that there are closed discrete subsets $D_n, n = 1, 2, \ldots,$ such that $D = \bigcup D_n$ is dense in $M$. Hence $|D| \leq m(\alpha)$ and so $|M| = |D| \leq m\alpha(\alpha)$.

Conversely, suppose $M$ is not $\alpha$-compact. Let $p \in \beta_\alpha(M) - M$. Put $\mathfrak{B} = p$ in Theorem 1. We have $|\mathfrak{B}| \geq m(\alpha)$, concluding the proof.

A space is said to be topologically complete if it is homeomorphic to a closed subspace of a product of metric spaces. The following is a generalization of the Katetov-Shirola theorem which states that a topologically complete space is real compact if and only if each closed discrete subspace has cardinality less than the first uncountable measurable cardinal. Real compact spaces are $\alpha$-compact spaces for $\alpha = \omega^+$. 

**Corollary 3.** A topologically complete space is $\alpha$-compact if and only if each closed discrete subset has cardinality less than $m(\alpha)$.

**Proof.** ($\Rightarrow$) This implication is trivial.

($\Leftarrow$) Let $X$ be a closed subspace of a product of metric spaces $\prod_{i \in I} M_i$ such that each closed discrete subset has cardinality less than $m(\alpha)$, we may assume that $X$ projects onto $M_i$ for $i \in I$. Then since a closed discrete subset of $M_i$ may be regarded as the projection of a closed discrete subset of $X$ we see that each closed discrete subset of $M_i$ has cardinality less than $m(\alpha)$ for each $i$ and so $X$ is $\alpha$-compact.

A family $\mathfrak{B}$ of zero sets of $X$ is called an $\alpha$-base if to each $Z \in Z(X)$ and each $x \in Z$ there is $\mathfrak{B} \subseteq \mathfrak{B}$ with

(i) $x \in \bigcap \mathfrak{B} \subseteq Z$,
(ii) $|\mathcal{B}| < m(\alpha)$,
(iii) $\cap \mathcal{B}' \in Z(X)$ for all $\mathcal{B}' \subseteq \mathcal{B}$.

**Example 1.** Let $(X, \tau)$ be a space and $w \leq \kappa \leq m(\alpha)$. Then $Z(X)$ is an $\alpha$-base for $(X, \tau(\kappa))$.

**Example 2.** (Cf. [4, Lemma 3.9].) The family $\bigcup \{\pi_i^{-1}(Z(X_i)) : i \in I\}$ where $\pi_i$ is the projection of $X = (\prod X_i)_\kappa$ to $X_i$ is an $\alpha$-base for $X$ if $\omega \leq \kappa \leq m(\alpha)$.

**Theorem 2.** Suppose $f: X \to Y$ is a continuous surjection such that $f^{-1}(Z(Y))$ is an $\alpha$-base for $X$. If $Y$ is $\alpha$-compact then so is $X$.

**Proof.** Suppose $X$ is not $\alpha$-compact, and let $p \in \beta_\alpha(X) - X$. Let $\tilde{f}(p) = y$ where $\tilde{f}$ is the Stone-Čech extension of $f$. Let $f(x) = y$. Then there is $Z \in Z(X)$ such that $x \in Z \not\subseteq p$. Since $f^{-1}(Z(Y))$ is an $\alpha$-base there is $\mathcal{B} \subseteq f^{-1}(Z(Y))$ such that $x \in \bigcap \mathcal{B} \subseteq Z, |\mathcal{B}| < m(\alpha)$ and $\mathcal{B}' \subseteq \mathcal{B} = \bigcap \mathcal{B}' \in Z(X)$. Clearly $\mathcal{B} \subseteq p$. There is $\tilde{Z} \in p$ such that $\tilde{Z} \cap (\bigcap \mathcal{B}) = \emptyset$. Hence the family $\{\tilde{Z}\} \cup \mathcal{B}$ is a subset of $p$ with empty intersection. By Theorem 1, $|\mathcal{B}| \geq m(\alpha)$ contrary to the assumption that $|\mathcal{B}| < m(\alpha)$. Hence $X$ is $\alpha$-compact as was to be proved.

**Corollary 4.** Let $(X, \tau)$ be $\alpha$-compact and let $\omega \leq \kappa \leq m(\alpha)$. Then $(X, \tau(\kappa))$ is $\alpha$-compact.

**Proof.** The identity map from $(X, \tau(\kappa))$ to $(X, \tau)$ satisfies the conditions of Theorem 2.

**Corollary 5.** If $\{X_i : i \in I\}$ is a collection of spaces with $|X_i| \geq 2$ for each $i$ then $(\prod X_i)_\kappa$ is $\alpha$-compact if and only if $X_i$ is $\alpha$-compact for each $i \in I$ and $\kappa \leq m(\alpha)$ or $|I| < m(\alpha)$.

**Proof.** Suppose $X = (\prod X_i)_\kappa$ is $\alpha$-compact. Then for each $i \in I$, $X_i$ can be considered as a closed subspace of $X$ and so is $\alpha$-compact. Let $J \subseteq I$ and $|J| < \kappa$. Let $D_j \subseteq X_j$ have exactly two elements for each $j \in J$ and let $D_j$ be a singleton subset of $X_j$ for $j \not\in J$. Then $\prod D_j$ is a closed discrete subset of $X$ and so has cardinality less than $m(\alpha)$. Hence $\kappa \leq m(\alpha)$ or $|I| < m(\alpha)$.

Conversely suppose $X_i$ is $\alpha$-compact for each $i \in I$ and $\kappa \leq m(\alpha)$ or $|I| < m(\alpha)$. Then $\prod_{i \in I} X_i$ is $\alpha$-compact and $Z(\prod X_i)$ is an $\alpha$-base for $(\prod X_i)_\kappa$. Hence $(\prod X_i)_\kappa$ is $\alpha$-compact by Theorem 2.

Corollary 5 is a generalization of Theorem 2.4 of Kato [3] and Corollary 4 is a generalization of 3.1 of the same paper. Kato deals with realcompact spaces. His method is quite different from ours.

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**References**


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