A SMOOTH SCISSORS CONGRUENCE PROBLEM

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Abstract. Classifying space techniques are used to solve a smooth version of the classical scissors congruence problem.

1. Introduction.

1.1 The classical problem [8]. Let $B$ be the abelian group generated by the set of polygons in the plane, modulo the subgroup generated by elements $P - \Sigma P_i$, where $P \big| P_i$ is a subdivision of a polygon $P$. Any subgroup $G$ of the group of affine motions of the plane acts on $B$. The problem is to compute the quotient group $H_0(G; B)$ of $B$ by the subgroup generated by elements $g \cdot b - b$, with $g \in G, b \in B$.

1.2 A smooth version. Our purpose is to state and solve a smooth version of the problem. Instead of polygons transforming under affine maps, we consider smooth curves transforming under diffeomorphisms.

The basic tool is a space $M$ (2.1) whose first singular integral homology group $H_1(M)$ is a smooth version of the group $B$. Diffeomorphisms of the plane act on $M$ and hence on $H_1(M)$. We employ a slight modification of a standard spectral sequence in our calculations.

1.3 Organization. §2 states the key definitions and results; the major proof is in §3. §4 contains the proof of a lemma, and §5 discusses the spectral sequence.

I would like to thank the referee for suggestions and for a simplification in the proof of Lemma 3.5.

2. Results. We require some definitions.

2.1 Definition. Let $M$ be the one-manifold of $C^\infty$ nonsingular curves in $\mathbb{R}^2$, defined as

$$M = \bigsqcup (a, b)_f \sim$$

where for each $C^\infty$ nonsingular embedding $f$ of an interval $(a, b)$ to $\mathbb{R}^2$ we take a copy $(a, b)_f$ of $(a, b)$, and where if $x \in (a, b)_f$ and $y \in (c, d)_g$ we set $x \sim y$ if and only if there exist neighborhoods $U$ of $x$ in $(a, b)_f$ and $V$ of $y$ in $(c, d)_g$ and a (not necessarily orientation preserving) diffeomorphism $h: U \to V$ such that $f|_U = g \circ h$.

$M$ is a one-dimensional $C^\infty$ nonorientable non-Hausdorff manifold; let $i: M \to \mathbb{R}^2$ denote the obvious immersion. If $g: U \to V$ is a diffeomorphism between open sets in $\mathbb{R}^2$, let $i^*g: i^{-1}U \to i^{-1}V$ denote the resulting diffeomorphism between the open
subsets $i^{-1}U$ and $i^{-1}V$ of $M$. Let $H_iM$ denote the first singular integral homology group of $M$.

2.2 Definition. Let $H_0(\Gamma^\infty; H_iM)$ (resp. $H_0(\Gamma^\Omega; H_iM)$) denote the quotient group of $H_iM$ by the subgroup generated by elements $(i^*g)_*b - b$, where $g: U \to V$ is an orientation preserving (resp. area and orientation preserving) $C^\infty$ diffeomorphism between open subsets of $\mathbb{R}^2$, and $b \in H_iM$ has support in $i^{-1}U$.

Our problem is to compute the groups just defined.

2.3 Example. The Figure 8 curve (with orientation given by the arrow in Figure 1) defines an element of $H_iM$. Here is one demonstration that this element is 0 in $H_0(\Gamma^\Omega; H_iM)$. The dotted curve indicates a part of $M$ used in each step.

\[ \text{Figure 1} \]

\[ \includegraphics[width=0.5\textwidth]{figure8.png} \]

2.4 Definition. (i) The winding maps $W: H_0(\Gamma^\infty; H_iM) \to \mathbb{Z}$, $W: H_0(\Gamma^\Omega; H_iM) \to \mathbb{Z}$. The tangent line field of $M$ defines a map from $M$ to $\mathbb{R}P^1$, and hence from $H_iM$ to $H_i\mathbb{R}P^1$. Picking an isomorphism of $H_i\mathbb{R}P^1$ with $\mathbb{Z}$ gives a map $H_iM \to \mathbb{Z}$, which pushes down to the maps $W$.

(ii) The area map $A: H_0(\Gamma^\Omega; H_iM) \to \mathbb{R}$: If $b \in H_iM$, let $A(b) = \int_{\{b\}} x \, dy$ (here $[b]$ denotes the one-current of $\mathbb{R}^2$ associated to $b$). $A(b)$ is the "algebraic area enclosed by $b$". $A$ pushes down to the map $A$.

2.5 Theorem. The maps $W: H_0(\Gamma^\infty; H_iM) \to \mathbb{Z}$ and $W \oplus A: H_0(\Gamma^\Omega; H_iM) \to \mathbb{Z} \oplus \mathbb{R}$ are isomorphisms.

2.6 Remark. We compare 2.5 with the classical result. Let $B$ (as in 1.1) be the abelian group generated by polygons in the plane, modulo the subgroup generated by subdivisions. Let $AG1$ and $AS1$ denote the group of orientation preserving affine maps of the plane and the subgroup of area and orientation preserving maps, respectively. Then $[B] H_0(AG1; B) = 0$, and area gives an isomorphism $A: H_0(AS1; B) \to \mathbb{R}$. There is no "winding map".

2.7 Remark. If in Definition 2.1 we glue the intervals $(a, b)_f$ together using orientation preserving diffeomorphisms $h$, we obtain a double cover $\tilde{M}$ of $M$, the one-manifold of $C^\infty$ oriented nonsingular curves in $\mathbb{R}^2$. There are winding maps $W: H_0(\Gamma^\Omega; H_i\tilde{M}) \to \mathbb{Z}$ and $W: H_0(\Gamma^\infty; H_i\tilde{M}) \to \mathbb{Z}$ defined via the tangent unit vector map from $M$ to $S^1$, and an area map $A: H_0(\Gamma^\Omega; H_i\tilde{M}) \to \mathbb{R}$. One can prove that $W: H_0(\Gamma^\infty; H_i\tilde{M}) \to \mathbb{Z}$ and $W \oplus A: H_0(\Gamma^\Omega; H_i\tilde{M}) \to \mathbb{Z} \oplus \mathbb{R}$ are isomorphisms.

3. Proof of 2.5. We shall prove that $W: H_0(\Gamma^\Omega; H_iM) \to \mathbb{Z} \oplus \mathbb{R}$ is an isomorphism. The proof for $W: H_0(\Gamma^\infty; H_iM) \to \mathbb{Z}$ is almost identical (see Remark 3.6).

Recall that a topological category is a small category whose sets of objects and morphisms are topologized such that the structure maps of the category are
continuous. The nerve of a topological category is a simplicial space; we use Segal’s “thick” realization (denoted ||·|| in [9, Appendix A]) to produce a classifying space functor |·| from topological categories to topological spaces.

3.1 Definition. Let \( \Gamma^0 \) be the topological category whose space of objects is \( \mathbb{R}^2 \), and whose space of morphisms, denoted \( \Gamma^0_i \), is the space of germs of \( C^\infty \) area and orientation preserving diffeomorphisms of \( \mathbb{R}^2 \), with the sheaf topology. Let \( D, R: \Gamma^0_i \rightarrow \mathbb{R}^2 \) denote the domain and range maps of \( \Gamma^0_i \).

The classifying space \( |\Gamma^0| \) is the “classifying space for \( C^\infty \) codimension 2 foliation, with a transverse orientation and area form”.

3.2 Definition. Let \( \Gamma^0 \setminus M \) be the topological category of the action \( \Gamma^0 \) on \( M \); the space of objects of \( \Gamma^0 \setminus M \) is \( M \), and the space of morphisms \( (\Gamma^0 \setminus M)_i \) of \( \Gamma^0 \setminus M \) is the pullback:

\[
\begin{array}{ccc}
(\Gamma^0 \setminus M)_i & \rightarrow & \Gamma^0_i \\
\downarrow D & & \downarrow D \\
M & \rightarrow & \mathbb{R}^2
\end{array}
\]

Let \( i: \Gamma^0 \setminus M \rightarrow \Gamma^0 \) denote the continuous functor covering the map \( i \).

Now we claim [2]

3.3 Proposition. There is a first quadrant spectral sequence \( E^*_p \), with differential \( d^n \) of bidegree \((-n, n - 1)\), which abuts to \( H_{p+1}(\Gamma^0 \setminus M) \) and such that \( E^0_p = H_p(\Gamma^0) \) and \( E^2_{p,1} = H_0(\Gamma^0; H_1 M) \).

The spectral sequence is discussed in §5. To apply it to the proof of 2.5 we need two lemmas.

3.4 Lemma [4, 2.6 and 6, Lemma 1]. \( H_1(\Gamma^0 \setminus M) = 0 \) and \( H_2(\Gamma^0 \setminus M) = \mathbb{Z} \oplus \mathbb{R} \).

3.5 Lemma. \( H_1(\Gamma^0 \setminus M) = \mathbb{Z}/2 \).

The proof of 3.5 is in §4.

Proof of Theorem 2.5. Let \( K \oplus C: \Gamma^0 \setminus M \rightarrow \mathbb{Z} \oplus \mathbb{R} \) be the isomorphism of Lemma 3.4. Considering the spectral sequence 3.3, 2.5 will follow from the facts that \( A \circ d^2 \circ C^{-1}: \mathbb{R} \rightarrow \mathbb{R} \) is an isomorphism and that the image of \( W \circ d^2 \circ K^{-1} \) is \( 2\mathbb{Z} \) (here \( d^2 \) is the differential for the \( E^2 \)-term). These facts will follow from an explicit description of \( d^2: H_2(\Gamma^0 \setminus M) \rightarrow H_0(\Gamma^0; H_1 M) \) for elements of \( H_2(\Gamma^0) \) represented by closed oriented two-manifolds with an area form.

Let \( X \) be such a two-manifold, and let \( [X] \in H_2(\Gamma^0) \) be the corresponding homology class; \( K[X] \) is the Euler characteristic of \( X \), and \( C[X] \) is the area of \( X \). To describe \( d^2[X] \), give a \( C^\infty \) cell decomposition \( X = \bigsqcup \sigma_i \) of \( X \) as in Figure 2. Each cell \( \sigma_i \) can be mapped to \( \mathbb{R}^2 \) by an orientation and area preserving diffeomorphism \( f_i \); the boundary of \( f_i \sigma_i \), with orientation inherited from \( X \), gives a cycle \( [\partial f_i \sigma_i] \in H_1(X) \). Then \( d^2[X] = \Sigma [\partial f_i \sigma_i] \) is well defined in \( H_0(\Gamma^0; H_1 M) \) and independent of the choice of \( C^\infty \) cell decomposition of \( X \).
Clearly $A \circ d^2 \circ C^{-1}$ is the identity, and a computation with $X = S^2$ shows that the image of $W \circ d^2 \circ K^{-1}$ is $2\mathbb{Z}$. This concludes the proof of 2.5.

3.6 Remark. The proof that $W: H_0(\Gamma^\infty; H_1 M) \to \mathbb{Z}$ is an isomorphism follows §3, except for the substitution of the following lemma for Lemma 3.4.

3.7 Lemma [4, Theorem 3]. $H_1 |\Gamma^\infty| = 0$ and $H_2 |\Gamma^\infty| = \mathbb{Z}$.

4. Proof of 3.5. The real line $\mathbb{R}$, embedded in $\mathbb{R}^2$ as the $x$-axis is a submanifold of $M$. Let $N$ be the discrete monoid of $\Gamma^0 \setminus M$-embeddings of the line; as a set

$$N = \{ s: \mathbb{R} \to (\Gamma^0 \setminus M) | D \circ s = \text{id} \text{ and } R \circ s(\mathbb{R}) \subseteq \mathbb{R} \}.$$ 

The translates of $\mathbb{R}$ by $(\Gamma^0 \setminus M)_s$ generate the topology of $M$, so by Theorem 1.2(ii) of [1] there is a weak homotopy equivalence $BN \to |\Gamma^0 \setminus M|$. Let us show that $\pi_1 BN = \mathbb{Z}/2$.

Let $K$ be the submonoid of $N$ consisting of elements which preserve the orientation of the line; it is not hard to see that the exact sequence $K \to N \to \mathbb{Z}/2$ gives a homotopy fibration $BK \to BN \to B\mathbb{Z}/2$. Since $\pi_2 B\mathbb{Z}/2 = 0$, 3.5 will follow when we show that $\pi_1 BK = 0$.

So we show that the homomorphic image of $K$ in any group is trivial. Now $K$ is generated by elements $k$ which are the identity section in some open set $U$ (after [7], 3.1). But for any $U$ there is an $m \in K$ such that $m(\mathbb{R}) \subseteq U$; therefore $km = m$ and $k$ must map to the identity of any group. So all of $K$ must map to the identity.

5. The spectral sequence 3.3. There is a spectral sequence for the action of a pseudogroup on a space, constructed in [2], which generalizes the spectral sequence for the action of a group on a space. The case at hand is an example of its application. We sketch the construction.

Let $C$ be the discrete category whose objects are contactible open subsets of $\mathbb{R}^2$, with morphisms area and orientation preserving embeddings between open sets. Note that (as in [8, §1]) there is a weak homotopy equivalence between $|C|$ and $|\Gamma^0|$.

Now recall the immersion $i: M \to \mathbb{R}^3$. Let $S_q$ denote the complex of abelian group valued functors of $C$, where for an open subset of $\mathbb{R}^2$, $S_q U = S_q (i^{-1} U)$, where $S_q$ is the usual singular $q$-chain functor. The spectral sequence for the complex $S_q$ of functors satisfies 3.3. In particular, $E^2_{p0} = H_p |\Gamma^0|$ because $i^{-1} U$ is connected if $U$ is connected.
BIBLIOGRAPHY


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