THE INDICES OF TORSION-FREE SUBGROUPS
OF FUCHSIAN GROUPS

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ABSTRACT. Elementary algebraic techniques are used to obtain the precise possible indices of torsion-free subgroups of finite index of finitely generated Fuchsian groups (and related groups).

Introduction. In a recent paper, Edmonds, Ewing and Kulkarni [3] announced the precise finite indices possible for a torsion-free subgroup of a finitely generated Fuchsian group (of orientation-preserving isometries of the hyperbolic plane). The proof (as described by the authors) is of a geometric-topological nature (they state that it contains "an involved sequence of explicit constructions of tesselations coupled with inductions and branched covering arguments"). In the present paper we give an elementary algebraic proof based on the permutation representations of the triangle groups, given by Fox [4] in his proof of Fenchel's conjecture, that every finitely generated Fuchsian group has a torsion-free subgroup of finite index. (Note that Fox's paper contains an error, corrected by Chau [1].)

Preliminaries. The finitely generated Fuchsian groups are characterized algebraically by the presentation

$$(1) \quad \Big\langle e_1,\ldots,e_r, a_1, b_1,\ldots,a_g, b_g, y_1,\ldots,y_s;$$
$$e_1^{m_1},\ldots,e_r^{m_r}, [a_1, b_1] \cdots [a_g, b_g] e_1 \cdots e_r y_1 \cdots y_s \Big\rangle,$$

where $r, g, s \geq 0$ but not all 0, $m_i > 1$ for $i = 1,\ldots,r$, and $[a, b]$ denotes the commutator $aba^{-1}b^{-1}$. The finitely generated non-orientation-preserving subgroups of the hyperbolic plane which contain no reflections are characterized algebraically by the presentation

$$(2) \quad \Big\langle e_1,\ldots,e_r, a_1,\ldots,a_g, y_1,\ldots,y_s; \; e_1^{m_1},\ldots,e_r^{m_r}, a_1^2 \cdots a_g^2 e_1 \cdots e_r y_1 \cdots y_s \Big\rangle,$$

where $r, g, s \geq 0$ but not all 0, $m_i > 1$ for $i = 1,\ldots,r$.

THEOREM 1 (EDMONDS, EWING AND KULKARNI). *Let $c = \mathrm{lcm}(m_1,\ldots,m_r)$ be the least common multiple of the exponents on $e_1,\ldots,e_r$ in (1) (or in (2)) [note: if $r = 0$ then $c = 1$], and let d be the remainder* $\mathrm{mod}\, 2$ *of 2^s times the number of even $m_1,\ldots,m_r$*

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having the same exponent on 2 *in their prime factorization as c. A torsion-free subgroup of finite index k exists in an infinite group with presentation* (1) (*or* (2)) *iff k is divisible by* $2^d c$.

In our proof of this theorem, the following two well-known results will be used repeatedly.

LEMMA 1. *A subgroup of finite index in a group with presentation* (1) *or* (2) *also has such a presentation.* $\square$

LEMMA 2. *Any element of finite order in an infinite group with presentation* (1) *or* (2) *is conjugate to a power of some* e_i. *Moreover, the order of* e_i *is* m_i. $\square$

For algebraic proofs of these lemmas see Hoare, Karrass and Solitar [5 and 6].

We shall also use the following result of Fox [4, cases II and III], which has an elementary algebraic proof (corrected in Chau [1]).

LEMMA 3. *If k, m and n are three positive integers with* $k \geqslant m, n$, *such that mnk is odd, or else mn is even and* $k \geqslant m + n$, *then there exist permutations* α *and* ω *in* S_k (*the symmetric group on k symbols*) *of orders m, n respectively such that* $\alpha\omega = (1 \cdots k)$. $\square$

We can now give a proof of the theorem. Throughout G will denote an infinite group with presentation (1) or (2) unless otherwise indicated. The following lemma establishes the necessity of the condition on the indices of torsion-free subgroups of finite index.

LEMMA 4. *A torsion-free subgroup of finite index in an infinite group with presentation* (1) *or* (2) *must have index divisible by c. Moreover, if* $d = 1$ *then the index must be divisible by* $2c$.

PROOF. If H is a torsion-free subgroup of G and J is a finite subgroup, then obviously J has trivial intersection with any conjugate of H. Hence each (J, H)-double-coset of G has $|J|$ H-cosets in it, and therefore $[G : H]$ is divisible by $|J|$. Thus $c | [G : H]$. Moreover, each element of J defines a regular (i.e., fixed-point-free) permutation of the H-cosets. Hence, if $[G : H]$ is an odd multiple of c, then each e_i with even order m_i having the same 2-exponent as c, defines a permutation with an odd number of even cycles, i.e., an odd permutation, while the other e_i define even permutations. But the product relation in (1) or (2) must define an even permutation (since it defines the identity), and moreover, a commutator or a square defines an even permutation. Hence, if $s = 0$, the product of the permutations corresponding to the e_i must be even. Thus, if $d = 1$ we get a contradiction of our assumption that $[G : H]$ is an odd multiple of c. $\square$

Sufficiency. An infinite torsion-free group with presentation (1) or (2) has $r = 0$, and clearly has an infinite cyclic quotient group and, therefore, torsion-free subgroups of any finite index. Hence, to prove the theorem, it suffices to show that any group G with presentation (1) or (2) has a torsion-free subgroup of index c if $d = 0$, and of index $2c$ if $d = 1$.

Certain cases on the values of r, s and g are relatively easy, and we first dispose of these.

When $r = 0$, G itself is torsion-free and $c = 1$, $d = 0$, and so this case is clear. When $s \geq 1$, G is a free product of cyclic groups, and may be mapped onto a cyclic group of order c, with e_i being mapped into the (c/m_i)th power of the generator while the other generators are mapped into 1, and so this case is also clear. Thus we may assume from now on that $s = 0$ and $r \geq 1$.

If $r = 1$ then, since G is infinite, $g \geq 1$. For (1) we can map G into $D_{2c} = \langle a, b; a^{2c}, b^2, (ab)^2 \rangle$, the dihedral group of order $4c$, with $e_1 \to a^{-4}$, $a_1 \to a^2$, $b_1 \to b$ if c is odd, and $e_1 \to a^{-2}$, $a_1 \to a$, $b_1 \to b$ if c is even; the other generators are mapped into 1. The preimage in G of $gp(b)$ under the described map will be torsion-free and have the required index. For (2) we can map G onto $Z_c = \langle a; a^c \rangle$, the cyclic group of order c, with $e_1 \to a^{-2}$, $a_1 \to a$, if c is odd, and map G onto $Z_{2c} = \langle a; a^{2c} \rangle$, the cyclic group of order $2c$, with $e_1 \to a^{-2}$, $a_1 \to a$, if c is even; the other generators are mapped into 1. The kernel of this map will be torsion-free and have the required index.

If $r = 2$ then, again, $g \geq 1$. If $d = 1$ or c is odd we reduce this case to a previous one by mapping G onto the group with presentation (1) or (2) and $r = 1$, $m_1 = c$, in effect by mapping the cyclic groups generated by e_1 and e_2 into their amalgamated direct product, a cyclic group of order c generated by $e_1 e_2$, and leaving the other generators as they are. If c is even and $d = 0$, then m_1 and m_2 both have the same highest power of two in them, say 2^p. Let $m_i = 2^p n_i$ for $i = 1, 2$, and let the cyclic group $gp(a)$ of order $2c$ be the direct product of the cyclic group $gp(u)$ of order 2^{p+1} and the cyclic group $gp(v)$ of odd order q. If G has presentation (1), then we map G into the dihedral group D_{2c} by sending $e_1 \to u^2 v^t$ and $e_2 \to u^2 v^x$, where $t = q/n_1$ and $x = q/n_2$. Then $e_1 e_2$ is mapped onto a fourth power of some element z in $gp(a)$. Hence, if $a_1 \to z^{-2}$, $b_1 \to b$, and the other generators are sent into 1, the image of G will be $gp(a^2, b)$ and the preimage of $gp(b)$ will be torsion-free with index c. If G has presentation (2) we map G into D_{2c} as above (but of course there is no b_1). The kernel of this mapping will be torsion-free with index c.

If $g \geq 1$, then by mapping all a_i, b_i into 1 and leaving the e_i's as they are, we clearly reduce the problem to that of finding torsion-free subgroups of the appropriate index in the finite and infinite groups having presentation (1) with *r ≥ 3* and *g = s = 0*, i.e.,

$$(3) \qquad G = \langle e_1, \ldots, e_r; e_1^{m_1}, \ldots, e_r^{m_r}, e_1 \cdots e_r \rangle, \qquad r \geq 3.$$

This group will be finite iff $r = 3$ and (m_1, m_2, m_3) is, in some order, one of the triples $(2, 2, n)$, $(2, 3, 3)$, $(2, 3, 4)$ or $(2, 3, 5)$. In each of these cases, the trivial subgroup 1 is torsion-free and has the appropriate index (which is the order of G) (see Coxeter and Moser [2]). We may therefore assume that group G in (3) is infinite.

We shall use induction on c.

Case 1. $d = 0$, c is odd, and the m_i are coprime in pairs.

Let p, p_2, p_3 be three distinct (odd) primes dividing three of the m_i, say, m_1, m_2, m_3, respectively, with $p \geq p_2, p_3$. By Lemma 3 there exist permutations α of order p_2

and ω of order p_3 in S_p, the symmetric group on p objects, such that $(1 \cdots p) = \alpha\omega$. The mapping $e_1 \to (1 \cdots p)$, $e_2 \to \omega^{-1}$, $e_3 \to \alpha^{-1}$, and $e_i \to 1$ for $i > 3$, defines a homomorphism ρ from G into S_p. Since the image ρG is transitive in S_p, the stabilizer of 1 in ρG has index p in ρG; hence, if H is the preimage of the stabilizer of 1 under ρ in G then $[G : H] = p$. Clearly, from Lemma 2, the orders of the elements of H divide those of G; moreover, some power of e_1 is in H iff p divides that power; also, some conjugate of e_2 and of e_3 is in H. It then follows from Lemmas 1 and 2 that H has presentation (1) or (2) with the lcm of the powers of its e_i equal to c/p. The torsion-free subgroup of H of index c/p, which exists by inductive hypothesis, will then have the required index c in G.

Case 2. $d = 0$, c is odd, the m_i are not coprime in pairs.

Let p be an odd prime which divides at least two of the m_i's, say $m_1, \ldots, m_k$, where $k \geqslant 2$, and no other m_i. Let $f_i = 0$ if $i > k$, $f_i = 1$ if $3 \leqslant i \leqslant k$, while $f_1 = 1$ and $f_2 = p - 1$ if p divides $k - 2$, but $f_1 = f_2 = (2 - k)/2 \bmod p$ if p does not divide $k - 2$. Clearly the mapping $e_i \to f_i$ defines a homomorphism ρ of G onto Z_p, the cyclic group of integers mod p. Some power of e_i for $1 \leqslant i \leqslant k$ is in H, the kernel of ρ, iff p divides that power. Hence, by Lemmas 1 and 2, H has a presentation (1) or (2) with the lcm of its m_i equal to c/p. The subgroup of H given by the inductive hypothesis is torsion-free with index c in G.

Case 3. $d = 0$ and c is even.

Suppose $c = 2^p q$ where q is odd. Since $d = 0$ and c is even, $p > 0$ and there are an even number of m_i's such that 2^p divides m_i, say $1 \leqslant i \leqslant k$, where k is even. Define $p(i)$ to be the highest power of 2 in m_i. We define a homomorphism ρ of G onto Z_{2^k}, the cyclic group of order 2^p, by mapping e_i to $2^p/2^{p(i)}$ for $2 \leqslant i \leqslant r$, and mapping e_1 to f where f is the odd number such that f added to the sum of the images of the other e_i is 0 mod 2^p. (Since $k - 1$ of the images of the other e_i's are 1 and the remaining images are even, it follows that f is indeed odd.) Clearly a power of some e_i is in H, the kernel of ρ, iff $2^{p(i)}$ divides that power; moreover, $[G : H] = 2^p$. Hence, by Lemmas 1 and 2, H has only elements of odd order, and in the presentation (1) or (2) for H, the lcm of its m_i is $q = c/2^p$. The torsion-free subgroup of H guaranteed by the inductive hypothesis then has index c in G.

Case 4. $d = 1$, at least 2 even m_i.

Suppose m_1, m_2 are even, where m_1 has the same 2-exponent as c. Consider the homomorphism ρ of G onto Z_2, the group of integers mod 2, given by $e_1 \to 1$, $e_2 \to 1$, and all other $e_i \to 0$. Clearly a power of e_1 or e_2 is in H, the kernel of ρ, iff the power is even; clearly $[G : H] = 2$. If m_1 is the only m_i with the same 2-exponent as c, then H has a presentation (1) or (2) in which the lcm of its m_i is $c/2$. Now H cannot have a torsion-free subgroup of index $c/2$, since then G would have a torsion-free subgroup of index c, contrary to Lemma 4. Hence, by inductive hypothesis H has a torsion-free subgroup of index $c = 2(c/2)$, and this has index $2c$ in G as required.

On the other hand, since $d = 1$, if some other m_i, in addition to m_1, has the same 2-exponent as c, then there are at least three such, so we may suppose the lcm of the presentation (1) or (2) for H is still c. Using a Reidemeister-Schreier rewriting

process to obtain a specific presentation for H with 1, e_1 as Schreier coset representative system (see Magnus, Karrass and Solitar [7]), and simplifying a bit, we obtain a presentation of type (1) with $g = s = 0$ and generators $f_1 = e_1^2$, $f_2 = e_2^2$, $f_3 = e_3, \ldots, f_r = e_r$, and $g_3 = e_1 e_3 e_1^{-1}, \ldots, g_r = e_1 e_r e_1^{-1}$. The exponents in the power relations are $m_1/2$, $m_2/2$, $m_3, \ldots, m_r$, $m_3, \ldots, m_r$. Hence, the number of generators with maximum 2-exponent is even, i.e. for H, $d = 0$ and c is even, Case 3 applies and therefore H has a torsion-free subgroup of index c which is of index $2c$ in G as required.

Case 5. m_1 even, m_i odd for $i > 1$, some pair of m_i not coprime.

Let p be an (odd) prime which divides at least two of the m_i's. Then we construct a homomorphism of G into Z_p as in Case 2, and a subgroup H with $[G : H] = p$ and with a presentation (1) or (2) in which the lcm is c/p. Now by inductive hypothesis H has a torsion-free subgroup of index c/p or $2c/p$. Hence G has a torsion-free subgroup of index $2c$, since one of index c is ruled out by Lemma 4.

Case 6. m_1 even, all the m_i coprime in pairs.

Let p, m be odd primes, $p > m$, dividing distinct odd m_i, and let $n = 2$. Then p, m, n satisfy the hypothesis of Lemma 3, and hence as in Case 1 we can find a subgroup H of G such that $[G : H] = p$ and H has a presentation (1) or (2) in which the lcm is c/p. By inductive hypothesis H has a torsion-free subgroup of index c/p or $2c/p$. Hence G has a torsion-free subgroup of index $2c$ as required, since one of index c is ruled out by Lemma 4.

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