A GLOSS ON A THEOREM OF FURSTENBERG

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Abstract. Certain refinements are offered for Furstenberg's ergodic-theoretic version of Szemeredi's theorem.

Furstenberg [1977] has proven a significant generalization of a theorem of Poincaré, which, with no real loss, can be formulated thus: If \( k \) is a positive integer and \( B_1, B_2, \ldots \) is a stationary sequence of events of positive probability in a countably additive probability space, then there is a \( k \)-progression, \( K \), such that 
\[
B_K = \bigcap_{k \in K} B_k
\]
has positive probability. (A \( k \)-progression is a set of \( k \) integers of the form \( \{a, a + b, a + 2b, \ldots, a + (k - 1)b\} \) with \( a > 0, b > 0 \).)

The present paper observes that neither the hypothesis of countable additivity nor of stationarity is needed. Moreover, the probability of \( B_K \) can be bounded from below by a \( \delta > 0 \) which depends only on \( k \) and \( p = P(B_1) \). These facts are immediate corollaries to:

**Theorem 1.** Let \( p > 0 \) and let \( k \) be a positive integer. Then there is a \( \delta > 0 \) and a positive integer \( n \) such that, for every \( n \)-tuple of events \( B_1, \ldots, B_n \) of average probability at least \( p \), there is a \( k \)-progression \( K \subset \{1, \ldots, n\} \) for which \( \bigcap B_i (i \in K) \) has probability at least \( \delta \).

This form of Furstenberg's theorem follows by an argument which he chose not to provide in [1977]. Indeed, it is a simple consequence of Szemeredi's theorem [1975] on the existence of arbitrarily long arithmetic sequences in each set of integers of positive density. But it is convenient first to provide a trivial lemma.

**Lemma 1.** Let \( B_1, \ldots, B_n \) be events of average probability at least \( p \) and let \( l \) be a positive integer less than \( n \). Then there is a subset \( X \) of \( \{1, \ldots, n\} \) of cardinality \( l \) such that
\[
P(\bigcap_{i \in X} B_i) \geq \left(p - \frac{l}{n}\right) \frac{n}{l}.
\]

**Proof of Lemma 1.** Let \( Y \) be the number of \( B \) that occur. Since \( Y \) is at most \( n \) on the event \( (Y \geq l) \) and is at most \( l - 1 \) on its complement, the following inequality (sharp) is easily obtained.
\[
P(Y \geq l) \geq \left(\frac{PY}{n} - \frac{l - 1}{n}\right) \left(1 - \frac{l - 1}{n}\right)^{-1}.
\]
(In (2), the precision (expectation) of $Y$ is designated by $PY$ as accords with a notational innovation of de Finetti.)

For the purposes of this note, this weaker inequality suffices:

$$P(Y > l) \geq \frac{PY}{n} - \frac{l}{n}.$$  

Plainly, the event $Y > l$ is the union of the events $\cap B_i (i \in X)$ as $X$ ranges over $[n]^l$, the subsets of $\{1, \ldots, n\}$ of cardinality $l$. Therefore,

$$P(Y > l) \leq \sum P(\cap B_i (i \in X)) \leq \left(\frac{n}{l}\right) \max P(\cap B_i (i \in X)),$$

as $X$ ranges over $[n]^l$. So, for some $X \in [n]^l$,

$$P(\cap B_i (i \in X)) \geq P(Y > l) \geq \left(\frac{PY}{n} - \frac{l}{n}\right) \left(\frac{n}{l}\right) \geq \left(p - \frac{l}{n}\right) \left(\frac{n}{l}\right),$$

where the second inequality obtains in view of (3), and the third by hypothesis.  

Let $\gamma_k(n)$ be the least integer $l$ such that, if $X$ is a subset of $\{1, \ldots, n\}$ of cardinality $l$, then $X$ includes a $k$-progression. Szemeredi [1975] has shown that $\gamma_k(n)/n$ converges to 0 as $n \to \infty$.

**Proof of Theorem 1.** By Szemeredi’s theorem, there is an $n = n(p, k)$ such that $\gamma_k(n) < np/2$. For $l = \gamma_k(n)$, let $\delta = p/2\gamma_k(n)$. That $(\delta, n)$ satisfies Theorem 1 can be verified, thus. Let $B_1, \ldots, B_n$ be events of average possibility at least $p$. By Lemma 1, there is an $X \subset \{1, \ldots, n\}$ of cardinality $l$ such that (1) holds. Since $l/n < p/2$, the right-hand side of (1) is at least $\delta$. So $\cap B_i (i \in X)$ has probability no less than $\delta$. Since $X$ is of cardinality $\gamma_k(n)$, $X$ includes a $k$-progression, $K$. Plainly, $\cap B_i (i \in K)$ includes $\cap B_i (i \in X)$. So it, too, has probability no less than $\delta$.

**References**


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