A NOTE ON COUNTABLY NORMED NUCLEAR SPACES

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Abstract. A modification of the Kömura-Kömura imbedding theorem is used to show that every countably normed nuclear space is isomorphic to a subspace of a nuclear Fréchet space with basis and a continuous norm. The space with basis can be chosen to be a quotient of \((s)\).

1. Introduction. By the famous Kömura-Kömura imbedding theorem [5] every nuclear Fréchet space is isomorphic to a subspace of \((s)^N\), where \((s)\) is the space of rapidly decreasing sequences. As a corollary, every nuclear Fréchet space is isomorphic to a subspace of a nuclear Fréchet space with basis. Since \((s)^N\) does not admit a continuous norm, we can ask to what extent this corollary holds for spaces with a continuous norm. We will show that a nuclear Fréchet space with a continuous norm is isomorphic to a subspace of a nuclear Fréchet space with basis and a continuous norm if (and only if) it is countably normed. (The concept of countably normedness was very important in constructing the first example of a nuclear Fréchet space without the bounded approximation property (see [1]).) Moreover, the space with basis can be chosen to be a quotient of \((s)\). The proof is a modification of the standard proof of the Kömura-Kömura theorem.

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2. Countably normed spaces. Let \(E\) be a Fréchet space which admits a continuous norm. The topology of \(E\) can then be defined by an increasing sequence \((\| \cdot \|_k)\) of norms (the index set is \(N = \{1, 2, \ldots\}\)). Let \(E_k\) denote \(E\) equipped with the norm \(\| \cdot \|_k\) only and let \(\hat{E}_k\) be the completion of \(E_k\). The identity mapping \(E_{k+1} \to E_k\) has a unique extension \(\phi_k: \hat{E}_{k+1} \to \hat{E}_k\) and this latter mapping is called canonical. The space \(E\) is said to be countably normed if the system \((\| \cdot \|_k)\) can be chosen in such a way that each \(\phi_k\) is injective.

To give an example of a countably normed space, assume that \(E\) has an absolute basis i.e. there is a sequence \((x_n)\) in \(E\) such that every \(x \in E\) has a unique absolutely converging expansion \(x = \sum_n x_n\), where \((\xi_n)\) is a sequence of scalars. Then \(E\) is isomorphic to the Köthe sequence space

\[
K(a) = K(a_n^k) = \left\{ (\xi_n) \mid (\xi_n) \big|_k = \sum_n |\xi_n| a_n^k < \infty \ \forall k \right\},
\]

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where $a_n^k = \|x_n\|_k$ (cf. [6, 10.1]). The topology of $K(a)$ is defined by the norms $| \cdot |_k$. The completions $(K(a)_k)^*$ can be isometrically identified with $l_1$ and then the canonical mapping $\phi_k : l_1 \to l_1$ is the diagonal transformation $(\xi_n)_n \mapsto ((a_n^k/a_{n+1}^k)\xi_n)_n$, which is clearly injective. Therefore $E$ is countably normed.

Consider now a nuclear Fréchet space $E$ which admits a continuous norm. The topology of $E$ can be defined by a sequence $\left(\| \cdot \|_k\right)$ of Hilbert norms, that is, $\|x\|_k = \langle x, x \rangle_k$, $x \in E$, where $\langle \cdot, \cdot \rangle_k$ is an inner product on $E$. The following result is due to Ed Dubinsky and the proof will be contained in [3].

**Theorem 1.** If a nuclear Fréchet space $E$ is countably normed, then the topology of $E$ can be defined by a sequence of Hilbert norms such that the canonical mappings $\phi_k : E_{k+1} \to E_k$ are injective.

Suppose finally that $(x_n)$ is a basis of $E$. Since $(x_n)$ is necessarily absolute [6, 10.2.1], $E$ can be identified with a Köthe space $K(a)$. By the Grothendieck-Pietsch nuclearity criterion [6, 6.1.2], for every $k$ there is $l$ with $(a_n^k/a_{n+1}^k) \in l_1$. Conversely, if the matrix $(a_n^k)$ with $0 < a_n^k \leq a_{n+1}^k$ satisfies this criterion, then the Köthe space $K(a)$ defined through (1) is a nuclear Fréchet space with a continuous norm and the sequence of coordinate vectors constitutes a basis. In particular, $(s) = K(n^k)$. The topology of such a nuclear Köthe space can also be defined by the sup-norms, $\| (\xi_n) \|_k,\infty = \sup_n |\xi_n| a_n^k$.

**3. An imbedding theorem.** We are now ready to prove the following characterization of countably normed nuclear spaces.

**Theorem 2.** Let $E$ be a nuclear Fréchet space which admits a continuous norm. Then the following two conditions are equivalent:

(i) $E$ is countably normed,

(ii) $E$ is isomorphic to a subspace of a nuclear Köthe space which admits a continuous norm.

Moreover, the Köthe space in (ii) can be chosen to be a quotient of $(s)$.

**Proof.** As explained in the introduction, a nuclear Köthe space with a continuous norm is countably normed. Since countably normedness is inherited by subspaces (e.g. [1, VI, 3.1.4]), the implication (ii) $\Rightarrow$ (i) is clear.

To prove (i) $\Rightarrow$ (ii) we choose a sequence $\left(\| \cdot \|_k\right)$ of Hilbert norms defining the topology of $E$ such that each canonical mapping $\phi_k : E_{k+1} \to E_k$ is injective (Theorem 1). Let $U_k = \{x \in E : \|x\|_k \leq 1\}$ and identify $(E_k)'$ with $E_k = \left\{ f \in E' : \|f\|_k = \sup_{x \in U_k} |\langle x, f \rangle| < \infty \right\}$.

Then $\phi_k' : E_k' \to E_{k+1}'$ is simply the inclusion mapping. As a Hilbert space, $E_{k+1}'$ is reflexive. Using this and the fact that $\phi_k : E_{k+1} \to E_k$ is injective, one sees easily that $\phi_k(E_k') = E_k'$ is dense in $E_{k+1}'$.

As in the standard proof of the Kōmura-Kōmura theorem (e.g. [6, 11.1.1]) we can construct in each $E_k'$ a sequence $(f_n^{(k)})_n$ of functionals with the following properties:
Now set \( g_n^{(1)} = f_n^{(1)}, n \in \mathbb{N} \), and using the fact that \( E_1 \) is dense in every \( E_k' \), choose \( g_n^{(k)} \in \mathbb{E}_k'^{l}, k \geq 2, n \in \mathbb{N} \), with
\[
\|f_n^{(k)} - g_n^{(k)}\|_{l} < 2^{-n}.
\]
In the construction of the desired Köthe space \( K(a) \) we will use two indices \( k \) and \( n \) to enumerate the coordinate basis vectors. First, set
\[
a_k^n = 2^k n^{2^{l}}, \quad k, n \in \mathbb{N}, l > k.
\]
Then choose \( a_k^n, a_k^{n-1}, \ldots, a_1^n \) so that
\[
1 > a_k^n \geq a_k^{n-1} \geq \ldots \geq a_1^n > 0, \quad k, n \in \mathbb{N},
\]
\[
\frac{a_k^{l+1}}{a_k^n} \geq \frac{a_k^{l+2}}{a_k^{l+1}}, \quad k, n \in \mathbb{N}, l \leq k,
\]
\[
a_k^n \leq \frac{1}{\|g_k^{(l)}\|_{l}}, \quad k, n \in \mathbb{N}, l \leq k.
\]
Note that (7) holds trivially for \( l > k \). Consequently, if \( K(a_k^n) = K(a) \) is nuclear, then it is also isomorphic to a quotient space of \((s)\) [2, Theorem 2.4]. But by (7), (5) and (6) for every \( l \geq 2,
\[
\sum_{k=1}^{l} \sum_{n=1}^{\infty} \frac{a_k^n}{a_k^{l+1}} = \sum_{k=1}^{l} \sum_{n=1}^{\infty} \frac{a_k^n}{a_k^{l+1}} + \sum_{k=1}^{l-1} \sum_{n=1}^{\infty} \frac{a_k^n}{a_k^{l+1}} < \infty.
\]
To embed \( E \) into \( K(a) \) we set \( Ax = (\langle x, g_k^{(n)} \rangle)_{k,n}, x \in E \). We have to show that \( Ax \in K(a), A: E \to K(a) \) is a continuous injection and that \( A^{-1}: A(E) \to E \) is also continuous.

Fix \( l \geq 2 \). Applying (3) to the sequences \( f_n^{(k)} \), \( k = 1, \ldots, l - 1 \), we can find an index \( p \geq l \) and a constant \( C \) such that
\[
\sup_{k \leq l, n} 2^k n^{2^{l}} |\langle x, f_n^{(k)} \rangle| \leq C \|x\|_p, \quad x \in E.
\]
From (5), (8) and (4) we then get for every \( x \in E \),
\[
|Ax|_\infty \leq \sup_{k,n} a_k^n |\langle x, g_k^{(n)} \rangle| \leq \sup_{k \leq l, n} a_k^n |\langle x, g_k^{(n)} \rangle| + \sup_{k > l, n} a_k^n |\langle x, g_k^{(n)} \rangle| \\
\leq \sup_{k \leq l, n} 2^k n^{2^{l}} |\langle x, g_k^{(n)} \rangle| + \|x\|_l \frac{1}{\|g_k^{(l)}\|_{l}} |\langle x, g_k^{(n)} \rangle| \\
\leq \sup_{k \leq l, n} 2^k n^{2^{l}} \|g_k^{(n)} - f_n^{(k)}\|_l \|x\|_k \\
+ \|x\|_l \leq C \|x\|_p,
\]
where \( C' = \sup_n n^{2^{l-2}} + C + 1 < \infty \).
Consequently, \( Ax \in K(a) \) and \( A: E \to K(a) \) is continuous. From (2) it follows that for every \( x \in E \),

\[
\| x \| = \sup_{f \in U^o} | \langle x, f \rangle | \leq \sup_n \left| \langle x, f_n(n) \rangle \right|.
\]

Further, since \( a_{n}^{l+1} > 1 \),

\[
\sup_n \left| \langle x, f_n(n) \rangle \right| \leq \sup_n \left| \langle x, f_n(n) - g_n(n) \rangle \right| + \sup_n \left| \langle x, g_n(n) \rangle \right|
\]

\[
\leq \sup_n \| f_n(n) - g_n(n) \| \| x \| + \sup_k, n \left| \langle x, g_n^{(k)} \rangle \right|
\]

\[
\leq \frac{1}{2} \| x \| + |Ax|_{l+1, \infty}.
\]

Thus, by (9) and (10) we have for every \( x \in E \),

\[
\| x \| \leq 2 |Ax|_{l+1, \infty}.
\]

Since \( l \) was arbitrary, this shows that \( A \) is injective and that \( A^{-1}: A(E) \to E \) is continuous. \( \Box \)

Finally we remark that it is not possible to find a single nuclear Fréchet space with basis and a continuous norm containing all countably normed nuclear spaces as subspaces. In fact, it was shown in [4] that not even any countable collection of nuclear Fréchet spaces with basis and a continuous norm contains all such spaces as subspaces.

**References**


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