A NOTE ON COUNTABLY NORMED NUCLEAR SPACES

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Abstract. A modification of the Kömura-Kömura imbedding theorem is used to show that every countably normed nuclear space is isomorphic to a subspace of a nuclear Fréchet space with basis and a continuous norm. The space with basis can be chosen to be a quotient of (s).

1. Introduction. By the famous Kömura-Kömura imbedding theorem [5] every nuclear Fréchet space is isomorphic to a subspace of (s)^N, where (s) is the space of rapidly decreasing sequences. As a corollary, every nuclear Fréchet space is isomorphic to a subspace of a nuclear Fréchet space with basis. Since (s)^N does not admit a continuous norm, we can ask to what extent this corollary holds for spaces with a continuous norm. We will show that a nuclear Fréchet space with a continuous norm is isomorphic to a subspace of a nuclear Fréchet space with basis and a continuous norm if (and only if) it is countably normed. (The concept of countably normedness was very important in constructing the first example of a nuclear Fréchet space without the bounded approximation property (see [1]).) Moreover, the space with basis can be chosen to be a quotient of (s). The proof is a modification of the standard proof of the Kömura-Kömura theorem.

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2. Countably normed spaces. Let E be a Fréchet space which admits a continuous norm. The topology of E can then be defined by an increasing sequence (|| · ||_k) of norms (the index set is N = {1, 2, ...}). Let E_k denote E equipped with the norm || · ||_k only and let  hat{E}_k be the completion of E_k. The identity mapping E_{k+1} → E_k has a unique extension phi_k:  hat{E}_{k+1} →  hat{E}_k and this latter mapping is called canonical. The space E is said to be countably normed if the system (|| · ||_k) can be chosen in such a way that each phi_k is injective.

To give an example of a countably normed space, assume that E has an absolute basis i.e. there is a sequence (x_n) in E such that every x ∈ E has a unique absolutely converging expansion x = Sigma_n xi x_n, where (xi) is a sequence of scalars. Then E is isomorphic to the Köthe sequence space

K(a) = K(a^k_n) = \{(xi) ||(xi)||_k = Sigma_n |xi| a^k_n < oo \forall k\}.

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where \( a_n^k = \|x_n\|_k \) (cf. [6, 10.1]). The topology of \( K(a) \) is defined by the norms \( |\cdot|_k \). The completions \( (K(a)_k)^\hat{*} \) can be isometrically identified with \( l_1 \) and then the canonical mapping \( \phi_k: l_1 \to l_1 \) is the diagonal transformation \( (\xi_n)_n \mapsto ((a_n^k/a_{n+1}^k)^*\xi_n)_n \) which is clearly injective. Therefore \( E \) is countably normed.

Consider now a nuclear Fréchet space \( E \) which admits a continuous norm. The topology of \( E \) can be defined by a sequence \( (\|\cdot\|_k) \) of Hilbert norms, that is, \( \|x\|_k = \langle x, x \rangle_k, x \in E \), where \( \langle \cdot, \cdot \rangle_k \) is an inner product on \( E \). The following result is due to Ed Dubinsky and the proof will be contained in [3].

**THEOREM 1.** If a nuclear Fréchet space \( E \) is countably normed, then the topology of \( E \) can be defined by a sequence of Hilbert norms such that the canonical mappings \( \phi_k: E_{k+1} \to E_k \) are injective.

Suppose finally that \( (x_n) \) is a basis of \( E \). Since \( (x_n) \) is necessarily absolute [6, 10.2.1], \( E \) can be identified with a Köthe space \( K(a) \). By the Grothendieck-Pietsch nuclearity criterion [6, 6.1.2], for every \( k \) there is \( l \) with \( (a_n^k/a_n^l) \in l_1 \). Conversely, if the matrix \( (a_n^k) \) with \( 0 < a_n^k \leq a_{n+1}^k \) satisfies this criterion, then the Köthe space \( K(a) \) defined through (1) is a nuclear Fréchet space with a continuous norm and the sequence of coordinate vectors constitutes a basis. In particular, \( (s) = K(n^k) \). The topology of such a nuclear Köthe space can also be defined by the sup-norms, \( |(\xi_n)|_{k, \infty} = \sup_n |\xi_n| a_n^k \).

3. **An imbedding theorem.** We are now ready to prove the following characterization of countably normed nuclear spaces.

**THEOREM 2.** Let \( E \) be a nuclear Fréchet space which admits a continuous norm. Then the following two conditions are equivalent:

(i) \( E \) is countably normed,

(ii) \( E \) is isomorphic to a subspace of a nuclear Köthe space which admits a continuous norm.

Moreover, the Köthe space in (ii) can be chosen to be a quotient of \( (s) \).

**PROOF.** As explained in the introduction, a nuclear Köthe space with a continuous norm is countably normed. Since countably normedness is inherited by subspaces (e.g. [1, VI, 3.1.4]), the implication (ii) \( \Rightarrow \) (i) is clear.

To prove (i) \( \Rightarrow \) (ii) we choose a sequence \( (\|\cdot\|_k) \) of Hilbert norms defining the topology of \( E \) such that each canonical mapping \( \phi_k: E_{k+1} \to E_k \) is injective (Theorem 1). Let \( U_k = \{x \in E||x||_k \leq 1\} \) and identify \( (E_k)' \) with \( E_k = \left\{ f \in E' ||f||_k = \sup_{x \in U_k} |\langle x, f \rangle| < \infty \right\} \).

Then \( \phi_k': E_k' \to E_{k+1}' \) is simply the inclusion mapping. As a Hilbert space, \( E_{k+1}' \) is reflexive. Using this and the fact that \( \phi_k: E_{k+1} \to E_k \) is injective, one sees easily that \( \phi_k'(E_k') = E_k' \) is dense in \( E_{k+1}' \).

As in the standard proof of the Kömura-Kōmura theorem (e.g. [6, 11.1.1]) we can construct in each \( E_k' \) a sequence \( (f_n^{(k)})_n \) of functionals with the following properties:

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Now set $g^{(1)}_n = f^{(1)}_n$, $n \in \mathbb{N}$, and using the fact that $E'_k$ is dense in every $E'_k$, choose $g^{(k)}_n \in E'_k$, $k \geq 2$, $n \in \mathbb{N}$, with
\[
\|f^{(k)}_n - g^{(k)}_n\|'_k < 2^{-n}.
\]
In the construction of the desired Köthe space $K(a)$ we will use two indices $k$ and $n$ to enumerate the coordinate basis vectors. First, set
\[
a^{(l)}_{kn} = 2^kn^2l, \quad k, n \in \mathbb{N}, l > k.
\]
Then choose $a^k_{kn}, a^{k-1}_{kn}, \ldots, a^1_{kn}$ so that
\[
1 > a^k_{kn} \geq a^{k-1}_{kn} \geq \ldots \geq a^1_{kn} > 0, \quad k, n \in \mathbb{N},
\]
and
\[
\frac{a^{l+1}_{kn}}{a^l_{kn}} \geq \frac{a^l_{kn}}{a^{l+1}_{kn}}, \quad k, n \in \mathbb{N}, l \leq k,
\]
so that
\[
a^{(l)}_{kn} \leq \frac{1}{\|g^{(l)}_k\|'_l}, \quad k, n \in \mathbb{N}, l \leq k.
\]
Note that (7) holds trivially for $l > k$. Consequently, if $K(a^{(l)}_{kn}) = K(a)$ is nuclear, then it is also isomorphic to a quotient space of $(s)$ [2, Theorem 2.4]. But by (7), (5) and (6) for every $l \geq 2$,
\[
\sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \frac{a^l_{kn}}{a^{l+1}_{kn}} = \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \frac{a^l_{kn}}{a^{l+1}_{kn}} + \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \frac{a^l_{kn}}{a^{l+1}_{kn}} \leq \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \frac{a^l_{kn}}{a^{l+1}_{kn}} + \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \frac{a^k_{kn}}{a^{k+1}_{kn}} \leq (l - 1) \sum_{n=1}^{\infty} \frac{1}{n^2} + \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} \frac{1}{2^kn^{2(l+1)}} < \infty.
\]
To imbed $E$ into $K(a)$ we set $Ax = ((x, g^{(l)}_{kn}))_{k, n}, x \in E$. We have to show that $Ax \in K(a), A: E \to K(a)$ is a continuous injection and that $A^{-1}: A(E) \to E$ is also continuous.

Fix $l \geq 2$. Applying (3) to the sequences $(f^{(k)}_n)_n$, $k = 1, \ldots, l - 1$, we can find an index $p \geq l$ and a constant $C$ such that
\[
\sup_{k < l, n} 2^{k}n^{2l} |\langle x, f^{(k)}_n \rangle| \leq \langle x, g^{(l)}_n \rangle \leq C\|x\|_p, \quad x \in E.
\]
From (5), (8) and (4) we then get for every $x \in E$,
\[
\|Ax\|_{l,\infty}^p = \sup_{k < l, n} a^l_{kn} |\langle x, g^{(l)}_n \rangle| \leq \sup_{k < l, n} a^l_{kn} |\langle x, f^{(l)}_n \rangle| + \sup_{k \geq l, n} a^l_{kn} |\langle x, g^{(l)}_n \rangle| \leq \sup_{k < l, n} 2^{k}n^{2l} |\langle x, g^{(l)}_n \rangle| + \sup_{k \geq l, n} \frac{1}{\|g^{(l)}_n\|'_l} |\langle x, g^{(l)}_n \rangle| \leq \sup_{k < l, n} 2^{k}n^{2l} \|g^{(k)}_n - f^{(k)}_n\|'_l |\langle x, g^{(l)}_n \rangle| + \sup_{k \geq l, n} 2^{k}n^{2l} |\langle x, f^{(l)}_n \rangle| + \|x\|_l \leq C\|x\|_p,
\]
where $C = \sup_n n^{2l}2^{-l-n} + C + 1 < \infty$. 

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Consequently, $Ax \in K(a)$ and $A: E \to K(a)$ is continuous. From (2) it follows that for every $x \in E$,

$$\|x\|_l = \sup_{f \in U^o} |\langle x, f \rangle| \leq \sup_n |\langle x, f_n^{(l)} \rangle|.$$  

Further, since $a_n^{l+1} > 1$,

$$\sup_n |\langle x, f_n^{(l)} \rangle| \leq \sup_n |\langle x, f_n^{(l)} - g_n^{(l)} \rangle| + \sup_n |\langle x, g_n^{(l)} \rangle|$$

$$\leq \sup_n \|f_n^{(l)} - g_n^{(l)}\|_l \|x\|_l + \sup_k a_k^{l+1} |\langle x, g_k^{(k)} \rangle|$$

$$\leq \frac{1}{2} \|x\|_l + \|Ax\|_{l+1, \infty}.$$  

Thus, by (9) and (10) we have for every $x \in E$,

$$\|x\|_l \leq 2 \|Ax\|_{l+1, \infty}.$$  

Since $l$ was arbitrary, this shows that $A$ is injective and that $A^{-1}: A(E) \to E$ is continuous. \[\Box\]

Finally we remark that it is not possible to find a single nuclear Fréchet space with basis and a continuous norm containing all countably normed nuclear spaces as subspaces. In fact, it was shown in [4] that not even any countable collection of nuclear Fréchet spaces with basis and a continuous norm contains all such spaces as subspaces.

References