ON PARTITIONS OF PLANE SETS INTO SIMPLE CLOSED CURVES. II

PAUL BANKSTON

Abstract. We answer some questions raised in [1]. In particular, we prove: (i) Let $F$ be a compact subset of the euclidean plane $E^2$ such that no component of $F$ separates $E^2$. Then $E^2 \setminus F$ can be partitioned into simple closed curves iff $F$ is nonempty and connected. (ii) Let $F \subseteq E^2$ be any subset which is not dense in $E^2$, and let $S$ be a partition of $E^2 \setminus F$ into simple closed curves. Then $S$ has the cardinality of the continuum. We also discuss an application of (i) above to the existence of flows in the plane.

Statement of results. This note is a sequel to [1], whose notation and terminology we follow faithfully. Throughout the paper, $F$ is a subset of the euclidean plane $E^2$, and $\mathcal{S}$ is an alleged partition of $E^2 \setminus F$ into simple closed curves (scc's) (i.e. $\mathcal{S}$ is a cover of $E^2 \setminus F$ by pairwise disjoint topological replicas of the unit circle). We are interested in two kinds of question: (i) (existential) what conditions on $F$ ensure or prohibit the existence of a partition $\mathcal{S}$; and (ii) (spectral) what are the relationships between $F$ and the set of cardinalities of possible partitions $\mathcal{S}$?

Existence questions are considered in [1, 2]. We summarize what we know: If the cardinality $|F|$ of $F$ is less than the continuum $c$, and if either the number of isolated points of $F$ or the number of cluster points of $F$ (in $E^2$) is finite, then $\mathcal{S}$ exists iff $|\mathcal{S}| = 1$. We conjecture that the conclusion is still valid under the weaker hypothesis "$|F| < c$"; however, the conclusion fails when the hypothesis is weakened further to "$F$ is totally disconnected", as is witnessed by a nice construction due to R. Fox [1, Theorem 12].

In [1] we also raise the question of when $\mathcal{S}$ exists for $F$ compact. This brings us to our first result.

1. Theorem. Let $F$ be a compact subset of $E^2$ such that no component of $F$ separates $E^2$. Then $E^2 \setminus F$ can be partitioned into scc's iff $F$ is nonempty and connected.

Questions of spectrum are considered in [1, 2, 4]; in particular, in [1] we ask: for which $F$ is it necessarily the case that $|\mathcal{S}| = c$ (if it exists at all)?

2. Theorem. Let $F$ be any subset of $E^2$ which is not dense in $E^2$, and let $\mathcal{S}$ be a partition of $E^2 \setminus F$ into scc's. Then $|\mathcal{S}| = c$. 

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The proof of Theorem 1 uses techniques from [1]. The proof of Theorem 2 is inspired by H. Cook’s proof [4] that every partition of $E^2$ into closed arcs must have cardinality $c$.

**Proof of Theorem 1.** Our first observation (due to the referee of [1]) is that the components of $F$, together with singleton points of $E^2 \setminus F$, form an uppersemicontinuous decomposition of $E^2$. By a theorem of R. L. Moore [6, p. 533] the corresponding quotient space is $\approx E^2$. In view of this it is easy to get the existence of $\mathcal{S}$ whenever $F$ is a nonempty continuum which fails to separate $E^2$, so it will suffice to prove

3. **Theorem.** Let $F \subseteq E^2$ be a compact totally disconnected subset of cardinality different from 1. Then $E^2 \setminus F$ cannot be partitioned into scc’s.

We can eliminate the case $F = \emptyset$ immediately [1, Theorem 1], so assume $|F| > 1$; and, for the sake of contradiction, let $\mathcal{S}$ be a partition of $E^2 \setminus F$ into scc’s. As in [1] we let $B(S)$ be the bounded component of $E^2 \setminus S$ for any scc $S$ and rely heavily on Schönflies’s theorem (i.e. $B(S) = E^2$). Also we will use the partial order $<$ on $\mathcal{S}$, given by $S_1 < S_2$ if $B(S_1) = B(S_1) \cup S_1 \subseteq B(S_2)$.

4. **Lemma.** If $F$ is totally disconnected and $\mathcal{M} \subseteq \mathcal{S}$ is a maximal chain, then $\bigcap \{B(S): S \in \mathcal{M}\}$ is a singleton subset of $F$.

**Proof.** This is proved in [1, Lemma 4]. □

Now for any $S \in \mathcal{S}$, $B(S) \cap F$ is a nonempty clopen subset of $F$, so for each clopen $G \subseteq F$ let $S_G = \{S \in \mathcal{S}: B(S) \cap F = G\}$. Then $\mathcal{S} = \bigcup \{S_G: G \subseteq F \text{ is clopen}\}$ is a (countable) union of pairwise disjoint subcollections, each of which is a chain under the $<$ -ordering. Let $U_G = \bigcup S_G$. Then the collection $\{U_G: G \subseteq F \text{ is clopen}\}$ is a cover of $E^2 \setminus F$ by pairwise disjoint sets (“annuli”). We will show that each $U_G$ is open. By a theorem of Kuratowski-Knaster [7], to the effect that $X$ separates $E^2$ only if a connected subset of $X$ separates $E^2$, we know that $E^2 \setminus F$ is connected. Hence, $U_G = \emptyset$ for all but one clopen $G \subseteq F$. We will show that, in fact, $S_F \neq \emptyset$, hence $\mathcal{S} = \mathcal{M}$. This will mean that $\mathcal{S}$ is a chain all of whose members enclose $F$, contradicting Lemma 4.

We will be done, therefore, once we prove the following two assertions.

5. **Lemma.** $S_F \neq \emptyset$.

**Proof.** Although we could argue as in the proof of [1, Lemma 9], the following approach (suggested by the referee) is more elementary.

View $E^2$ as $S^2 \setminus \{p\}$ (i.e. the two-sphere minus the point at infinity); and for each scc $S \subseteq E^2$ let $U(S)$ be the complement of $B(S) \cup S$ in $S^2$. Since the collection of $<$ -maximal elements of $\mathcal{S}$ is at most countable and each chain in $\mathcal{S}$ without a $<$ -maximal element has countable cofinality, we can find a countable collection $S_1, S_2, \ldots$ in $\mathcal{S}$ which includes the $<$ -maximal elements and such that $\bigcup_{n=1}^{\infty} B(S_n) = \bigcup \{B(S): S \in \mathcal{S}\}$. For $m = 1, 2, \ldots$, let $C_m = \bigcap_{n=1}^{m} \overline{U(S_n)}$. Then $C_1, C_2, \ldots$ is a decreasing chain of continua containing $p$. Suppose $C_m \cap F \neq \emptyset$ for each $m$. Since $F$ is compact, we have that $C = \bigcap_{m=1}^{\infty} C_m$ is a continuum which intersects $F$ and...
contains $p$. By a theorem of Sierpiński [6, p. 173; 1, Lemma 8(ii)], to the effect that no locally compact connected Hausdorff space can be partitioned into countably many proper compact subsets, $C$ must contain a point $x$ not in $F \cup \{p\} \cup \bigcup_{n=1}^{\infty} S_n$. But $x \in S$ for some $S \in \mathcal{S}$, and $S$ cannot be $<\text{-maximal}$. Thus $x \in B(S_n)$ for some $n$, a contradiction. Thus for some $m$, $F \cap C_m = \emptyset$; hence $F \subseteq \bigcup_{n=1}^{m} B(S_n)$.

For each $x \in F$ let $S_x = \{S \in \mathcal{S}: x \in B(S)\}$. By [1, Theorem 1], $\mathcal{S} = \bigcup_{x \in F} S_x$, and each $S_x$ is a chain. Let $G_x = \bigcup \{B(S): S \in \mathcal{S}_x\}$. By the above argument, $x \in G_x \subseteq H_x$ is a closed disk if $S_x$ has a $<\text{-maximal}$ element, and $G_x$ is a chain union of open disks if $S_x$ has no $<\text{-maximal}$ element. Furthermore, the collection $\{G_x: x \in F\}$ is a finite partition of $E^2$. But the complement of a finite nontrivial union of disjoint closed disks is multiply connected. Hence $G_x = E^2$ for each $x \in F$ and $F \subseteq B(S)$ for some $S \in \mathcal{S}$. $\square$


Proof. Let $G \subseteq F$ be clopen and assume $S_G \neq \emptyset$. Since $S_G$ is a chain it will suffice to show that $S_G$ has no $<\text{-minimal}$ or $<\text{-maximal}$ element. Let $S \in S_G$. Since $B(S) \approx E^2$ and $G$ is compact, we can apply Lemma 5 relativized to $B(S)$. Thus, there is a scc $S' \in S$ with $G \subseteq B(S') \subseteq B(S') \subseteq B(S)$. Clearly $S' \in S_G$, so $S_G$ has no $<\text{-minimal}$ element.

To see that $S_G$ has no $<\text{-maximal}$ element, we “exchange” the point $p$ at infinity for any element of $G$. ($G$ is nonempty.) The ordering $<$ is reversed and we apply the above argument to the compact set $(F \setminus G) \cup \{p\}$. This finishes the proof of the lemma, and hence of Theorem 1. $\square$

Proof of Theorem 2. Suppose $F$ is a subset of $E^2$ which is not dense in $E^2$, and let $\mathcal{S}$ be a partition of $E^2 \setminus F$ into scc’s. Let $D$ be a standard open disk with boundary circle $C$ such that $D \cap F = \emptyset$. Then no $S \in \mathcal{S}$ lies in $D$ [1, Theorem 1]; so for each $S \in \mathcal{S}$, $S \cap D$ is a countable disjoint union of open arcs with distinct endpoints on $C$. Since these arcs form a partition of $D$ and each such arc is a subarc of a member of $\mathcal{S}$, it will suffice to show that it takes $c$ arcs to do the job. Let $A$ be one of the arcs and let $D_A$ be a component ($\approx E^2$) of $D \setminus A$. We show that $c$ arcs are necessary to fill $D_A$ by proving the following.

7. Lemma. Let $[0, 1]^2$ denote the closed unit square and let $\mathcal{E}$ be a partition of the open unit square $\mathcal{O} = (0, 1)^2$ by open arcs (i.e. homeomorphs of $(0, 1)$), each with distinct endpoints on $[0, 1] \times \{0\}$. Then $|\mathcal{E}| = c$.

Proof. The following argument is similar to that given by H. Cook in [4] to show that $E^2$ cannot be partitioned into $< c$ closed arcs.

By the Baire Category Theorem applied to $(0, 1)^2$ (each $A \in \mathcal{E}$ is closed as well as nowhere dense in $(0, 1)^2$), we know that $\mathcal{E}$ is uncountable; hence there is a real $\delta > 0$ and an uncountable $\mathcal{E}_0 \subseteq \mathcal{E}$ such that the endpoints of each $A \in \mathcal{E}_0$ have a distance apart of at least $\delta$. For $A \in \mathcal{E}$, let $l(A)$ (resp. $r(A)$) denote the left (resp. right) endpoint of $A$, and let $B(A)$ denote the region bounded by $A$ and $[l(A), r(A)] \times \{0\}$. We order $\mathcal{E}$ by writing $A_1 \prec A_2$ if $B(A_1) \subseteq B(A_2)$ and $A_1 \neq A_2$. Now suppose $n$ is any whole number such that $n\delta > 1$. Then $\mathcal{E}_0$ has at most $n$ maximal chains under
<. (This follows from the fact that < is a tree ordering; hence, if there were > n maximal chains in \( \mathcal{A}_0 \), then there would be > n arcs \( A \in \mathcal{A}_0 \) such that the regions \( B(A) \) are pairwise disjoint. Since their endpoints have a distance apart of \( \geq \delta \), this is impossible.) Thus there is an uncountable \( \mathcal{A}_1 \subseteq \mathcal{A}_0 \) which is a chain under the < -order. Assume \( |\mathcal{A}_1| < c \), and let \( \mathcal{C} \) denote the space of subcontinua of \([0, 1]^2\). Under the well-known Hausdorff metric, \( \mathcal{C} \) is a compact metric space. Let \( \mathcal{A}_1 \) denote the closure of \( \mathcal{A}_1 \) in \( \mathcal{C} \). Then \( |\mathcal{A}_1| = c \). (To see this we use the facts that \( \mathcal{C} \) is hereditarily Lindelöf, and in such spaces scattered subsets are countable. Since \( \mathcal{A}_1 \) is uncountable it is not scattered; hence, it has a nonempty closed subset without isolated points. This subset, being also compact metric, contains Cantor sets.)

Since we are assuming \( |\mathcal{A}_1| < c \), we have \( |\mathcal{A}_1 \setminus \mathcal{A}_1| = c \). Also, since \( \mathcal{A}_1 \) is a chain, each element of \( \mathcal{A}_1 \setminus \mathcal{A}_1 \) is a limit either from above or below of distinct arcs in \( \mathcal{A}_1 \), say \( B = \lim_{n \to \infty} A_n \) where \( A_{n+1} < A_n \). Hence, \( B \) intersects at most one arc in \( \mathcal{A}_1 \) and at most one other continuum in \( \mathcal{A}_1 \). Let
\[
 r = \inf \{r(A): A \in \mathcal{A}_1 \}, \quad l = \sup \{l(A): A \in \mathcal{A}_1 \},
\]
and let \( (a, 0) \) be the midpoint of the segment \([l, r] \times \{0\} \) \( (r - l \geq \delta) \). Let \( L \) be the vertical segment \( \{a\} \times [0, 1] \). Then each \( B \in \mathcal{A}_1 \) intersects both \([0, a] \times \{0\}\) and \([a, 1] \times \{0\}\). Let \( f: (\mathcal{A}_1 \setminus \mathcal{A}_1) \to L \) take a continuum \( B \) to a point of \( B \cap (L \setminus \{(a, 0)\}) \). Then the image \( f(\mathcal{A}_1 \setminus \mathcal{A}_1) \) has cardinality \( c \) since the fibers of \( f \) have at most two elements.

Finally, it is plain that if \( x_1, x_2, x_3 \) are three distinct points of \( f(\mathcal{A}_1 \setminus \mathcal{A}_1) \) then some arc of \( \mathcal{A}_1 \) separates two of them in \([0, 1]^2\). Thus no member of the original family \( \mathcal{A} \) can contain more than two points of \( L \). Since every point of \( f(\mathcal{A}_1 \setminus \mathcal{A}_1) \) lies on exactly one arc in \( \mathcal{A}_1 \), this says that \( |\mathcal{A}_1| = c \). \( \square \)

8. Remark. Theorem 2 contrasts nicely with the fact \([5, 9]\) that, under hypotheses consistent with the usual axioms of set theory, \( E^2 \) can be covered by < \( c \) (possibly overlapping) scc's.

An application to the theory of flows. In this section we follow the terminology found in Beck [3]. A flow in \( E^2 \) is a continuous surjection \( f: E^1 \times E^2 \to E^2 \) with the "group property" \( f(s + t, x) = f(s, f(t, x)) \). We define a flow to be periodic if for each \( x \in E^2 \), either \( x \) is a fixed point of \( f \) (i.e. \( f(t, x) = x \) for all \( t \in E^1 \)) or \( p_f(x) = \inf \{t > 0: f(t, x) = x \} \) is finite and positive. It is an easy exercise (see [3]) to show that \( x \) is a fixed point iff \( p_f(x) = 0 \); and the orbit of \( x \), \( \{f(t, x): t \in E^1\} \), is a scc iff 0 < \( p_f(x) < \infty \).

9. Theorem (Beck [3, Corollary 6.20]). Let \( F \subseteq E^2 \) be a compact set whose complement is homeomorphic with the open annulus \( \{x \in E^2: 1 < |x| < 2\} \). Then there exists a periodic flow on \( E^2 \) whose set of fixed points is \( F \).

Let \( F \subseteq E^2 \) be nonempty and compact, such that both \( F \) and \( E^2 \setminus F \) are connected. Letting \( \mathcal{A}_F \) be the uppersemicontinuous decomposition of \( E^2 \) into \( F \) together with singletons of \( E^2 \setminus F \), we have immediately from Moore's theorem (i.e. \( E^2 / \mathcal{A}_F \approx E^2 \)) that \( E^2 \setminus F \) is homeomorphic with an open annulus. Putting Theorem 9 and our Theorem 1 together we have the following existence theorem for flows.
10. **Theorem.** Let $F \subseteq E^2$ be compact, no component of which separates $E^2$. The following are equivalent:

(i) $F$ is nonempty and connected.

(ii) There exists a periodic flow on $E^2$ whose fixed point set is $F$.

11. **Remark.** The (ii) $\Rightarrow$ (i) direction is a very weak corollary of Theorem 1, which in effect offers a “static” (rather than “dynamic”) argument for the nonexistence of flows.

**References**


**Department of Mathematics, Statistics and Computer Science, Marquette University, Milwaukee, Wisconsin 53233**