PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 89, Number 3, November 1983

EXTENDING FAMILIES OF DISJOINT ZERO SETS

C. E. AULL

ABSTRACT. The z-cellularity $z(X)$ of a space X is defined as

$$z(X) = \sup\{|Z|: Z \subset Z(X)\}$$

where $Z(X)$ is the family of zero sets of X. It is proved using CH that a Tychonoff space S is TC^*-embedded in every Tychonoff space it is C-embedded in iff $z(S) \leq c$. A space S is defined to be TC^*-embedded in a space X if any disjoint family of zero sets of S can be extended to a family of disjoint zero sets of X. Similar theorems are proved for C^*-embedding when S is a P-space or the zero sets have the Isiwata property.

1. Introduction. If a set S is dense and C-embedded in a Tychonoff space X the closure of a zero set of S in X is a zero set of X [**6**]. Hence any family of disjoint zero sets of S may be extended to a family of disjoint zero sets of X. A set A of S is said to be extended to a set $E(A)$ of X if $E(A) \cap S = A$.

In this paper, we investigate the disjoint extension of families of disjoint zero sets of a set to a space, some families with restricted cardinals and some with restrictions to regular zero sets. Unless noted, definitions and terminology will be found in [**6**].

THEOREM 1. *A space S is C^*-embedded in a space X iff any denumerable family of disjoint zero sets may be extended disjointly to a family of zero sets of X.*

PROOF. Let $\{Z_n\}$ be a disjoint denumerable family of zero sets of X. Let H_{nk} be a zero set extension of Z_n to X disjoint from H_{kn}, $k \neq n$, guaranteed by the C^*-embedding of S in X. Then the family $\{H_n: H_n = \bigcap\{H_{nk}: k \neq n\}\}$ is the desired family of disjoint extensions of $\{Z_n\}$.

DEFINITION 1. A set S is TC^*-embedded (T_z-embedded) in a space X if any disjoint family of zero (cozero) sets of S can be extended to a disjoint family of zero (cozero) sets of X.

In [**1**], T_z-embeddings were studied.

It is clear that any C^*-embedded zero set is TC^*-embedded and if every subset of a space is TC^*-embedded, then the space is perfectly normal and extremally disconnected.

2. C-embeddings and TC^*-embeddings. We can improve Theorem 1 with respect to TC^*-embeddings when the embedded set is C-embedded and dense as mentioned earlier. More generally we have the following theorem.

THEOREM 2. *Let S be C-embedded in a Tychonoff space X. Then ω_1 disjoint zero sets in S may be extended to disjoint zero sets of X.*

Received by the editors August 9, 1982 and, in revised form, February 7, 1983.

1980 *Mathematics Subject Classification*. Primary 54C50, 54C45; Secondary 54D60, 54G05, 54G10, 54G20.

PROOF. We order the ω_1 zero sets $\{Z_\alpha\}$ by putting them into one-to-one correspondence with the countable ordinals. Let $E(Z_\alpha)$ be a zero set extension of Z_α for $\alpha < \omega_1$. By means of Lemma 2, we construct a zero set Z_α^β of X such that $Z_\alpha \subset Z_\alpha^\beta$ and $Z_\alpha^\beta \cap E(Z_\beta) = \varnothing$ for each $\beta < \alpha$. The family $\{H_\alpha\}$, $H_\alpha = \bigcap_\beta \{Z_\alpha^\beta \colon \beta < \alpha\} \cap E(Z_\alpha)$ is a family of disjoint zero set extensions of $\{Z_\alpha\}$.

LEMMA 2. *A z-embedded set S is C-embedded in a space X iff for H a zero set of S and Z a zero set of X, $H \cap Z = \varnothing$, H and Z are completely separated in X.*

PROOF. *Sufficiency.* Since S is z-embedded and completely separated from any disjoint zero set of X, S is C-embedded in X [**3**].

Necessity. Let $f \in C(X)$ and $h \in C(S)$ such that $Z = f^{-1}(0)$ and $H = h^{-1}(0)$. Then $|f| + |h|$ is continuous on S and can be extended to a positive continuous function g on X. Set $h' = g - |f|$. Then $h' = |h|$ on S. Since $g > 0$, $h' > 0$ on Z. So $h'^{-1}(0)$ and Z are disjoint zero sets of X containing H and Z respectively.

REMARK 1. It ic clear from the above proof that C-embedding of a set $S \subset X$ is equivalent to C^*-embedding and every positive $f \in C^*(X)$ having a positive continuous extension. This is probably known.

DEFINITION 2. (z-cellularity) $z(X) = \sup\{|\mathcal{Z}| \colon \mathcal{Z} \subset \mathcal{Z}(X),\ \mathcal{Z}$ a disjoint family$\}$ where $\mathcal{Z}(X)$ is the family of all zero sets of X.

COROLLARY 2. *If X is almost compact* [**6**] *with $z(X) = \omega_1$, then X is TC^*-embedded in every Tychonoff space X is embedded in.*

The necessity of the above conditions follows from the next theorem.

THEOREM 3 (CH). *A Tychonoff space X is TC^*-embedded in every Tychonoff space it is C-embedded (C^*-embedded) in iff (only if) $z(X) \leq c$.*

PROOF. Sufficiency of the C-embedding result follows from Theorem 2. Necessity is obtained by embedding νX in the product of real lines, in which X will be C-embedded in and by Engelking's [**4**, p. 295] result that $z(X) \leq c$ for the product of regular separable spaces. Thus if $z(X) > c$, there is a Tychonoff space in which X is C-embedded in but not TC^*-embedded in. The C^*-embedding result follows.

REMARK 2. We cannot replace C-embedding by C^*-embedding in the sufficiency argument as Ralph Fox has shown that R is not TC^*-embedded in βR.

COROLLARY 3 (CH). *A Tychonoff space X is TC^*-embedded in every Tychonoff space it is embedded in iff X is almost compact and $z(X) \leq c$.*

2. C^*-embeddings and TC^*-embeddings.

THEOREM 4. *If a family of ω_1 disjoint zero sets of a Tychonoff space S may be extended disjointly to βS, then this family of zero sets of S may be disjointly extended to any Tychonoff space S is C^*-embedded in.*

PROOF. If S is C^*-embedded in X, $\beta S \subset \beta X$ and βS is C-embedded in βX. By Theorem 2 ω_1 disjoint zero sets of βS may be extended to βX. The result follows.

COROLLARY 4 (CH). *Let S be Tychonoff and TC^*-embedded in βS. Then S is TC^*-embedded in every space it is C^*-embedded in iff $z(S) \leq c$.*

It is not unusual to have a set S TC^*-embedded in βS even when there are families of disjoint zero sets of S of large cardinality. This is always true when S is pseudocompact or discrete. In these cases, by Theorem 4, any family of ω_1 disjoint zero sets may be extended disjointly to a family of zero sets of any Tychonoff space in which S is C^*-embedded in. The next two theorems are modifications of these results.

THEOREM 5. *Let $\{z_\alpha\}$ be a family of disjoint zero sets of cardinality ω_1 of a set S such that for $f \in C^*(S)$ such that $Z(f) \cap Z_\alpha = \emptyset$, $|f(z)| \geq \epsilon > 0$ for $z \in Z_\alpha$. Then $\{Z_\alpha\}$ may be extended disjointly to any space in which S is C^*-embedded.*

PROOF. Let $E(Z_\alpha)$ be a zero set extension of Z_α to X. Thus as in Lemma 2, $E(Z_\alpha)$ is completely separated in X from Z_β for $\beta \neq \alpha$. The proof is completed as in Theorem 2.

The property referred to at the end of the first sentence of the statement of the theorem was discovered by Isiwata [**7**] to be satisfied by all zero sets of a pseudocompact space. We will refer to the above property as the Isiwata property. By modifying the proof of Theorem 5, one may prove the following result.

THEOREM 5A. *let $S \subset X$, X Tychonoff, have at most ω_1 disjoint zero sets and let every disjoint family of zero sets, none of which has the Isiwata property be denumerable. Then S is TC^*-embedded in X if S is C^*-embedded in X.*

DEFINITION 3 (COMPARE [**2**]). A weak 0_z-space is a space where the closure of a cozero set is a zero set.

We will call a zero set that is also a regular closed set a regular zero set.

THEOREM 6. *Let βS be weak 0_z. If $\{Z_\alpha\}$ is a disjoint family of regular zero sets (of cardinality ω_1) then $\{Z_\alpha\}$ may be extended disjointly to a family of zero sets of βS (of X Tychonoff where S is C^*-embedded in X).*

PROOF. Since βS is weak 0_z, S is weak 0_z. So the interior of Z_α is a cozero set. From [**1**], $\{\operatorname{int} Z_\alpha\}$ has a family of disjoint cozero extensions $\{C_\alpha\}$ in βS. It is immediate that in βS, $\overline{C}_\alpha = \overline{Z}_\alpha$ so that $\overline{Z}_\alpha$ is a zero set of βS. By the C^*-embedding of S in βS, these zero sets are necessarily disjoint. An application of Theorem 4 completes the proof.

An important class of spaces that satisfy the conditions of the theorem are basically disconnected spaces.

COROLLARY 6. *Let S be a P-space. Then S is TC^*-embedded in βS and any family of ω_1 disjoint zero sets of S may be extended disjointly to any Tychonoff space S is C^*-embedded in.*

PROOF. Since S is a P-space, βS is basically disconnected and each zero set of S is a regular closed set.

4. Some examples. The author is indebted to E. van Douwen for pointing out the following theorem of Juhasz which will be useful in analyzing examples.

THEOREM A (JUHASZ [**8**]). *If X is compact (even if X is the union of G_δ-sets of some compactification, i.e. X is of point-countable type) and S has cellularity m, m infinite, then X has at most 2^m disjoint G_δ-sets.*

In the above theorem, we can replace G_δ-sets by zero sets and compact by pseudocompact sets based on Theorem 2.

We have already noted R. Fox's result that R is not TC^*-embedded in βR. However as all disjoint families of regular zero sets are countable, these families are disjointly extendable. By Theorem 5, using CH, all disjoint families of compact zero sets of R are disjointly extendable.

The following example of R. Pol, pointed out to the author by R. Hodel, shows that even in the case of dense C^*-embedding we may be limited in the cardinality of the number of disjoint zero sets that are extended in case S is not pseudocompact or a P-space.

EXAMPLE 1 (R. POL [**9**]). The space S is Tychonoff, satisfies c.c.c., has countable pseudocharacter and has cardinality 2^c. Thus S has 2^c disjoint zero sets. Furthermore βS satisfies c.c.c; so by Theorem A the z-cellularity of βS is a most c. So S is not TC^*-embedded in βS.

EXAMPLE 2. Let X be the absolute of S; then βX is the absolute of βS [**11**] and since c.c.c. is preserved under taking absolutes [**11**], X satisfies c.c.c. and since there is a continuous map from X to S, $z(x) \geq 2^c$. By Theorem A and the fact that βX satisfies c.c.c., the z-cellularity of βX is at most c. Thus X is not TC^*-embedded in βX and X is extremally disconnected.

EXAMPLE 3. Since $\beta N \sim N$ is a zero set of βN, $\beta N \sim N$ is TC^*-embedded in βN and is not T_z-embedded in βN. Even though C^*-embedding implies z-embedding TC^*-embedding does not imply T_z-embedding.

Blair [**2**] has shown that if a weakly perfectly normal space is not realcompact one can add a point of $\nu X \sim X$ and obtain a weakly perfectly normal space that is not perfectly normal (the additional point is not a G_δ in the space). We will call this space $W(X)$. It is clear that if X is normal and hereditarily extremally disconnected then so will $W(X)$ be.

EXAMPLE 4 (BLAIR [**2**]). If D is a discrete space of measurable cardinality $W(D)$ is a hereditarily extremally disconnected normal space that is not perfectly normal. Furthermore it is easily shown that every subset of $W(D)$ is TC^*-embedded so that every subset of a space may be TC^*-embedded without the space being perfectly normal.

Below we have another example of this where we use the set-theoretic axiom club.

EXAMPLE 5 (WAGE [**10**]). The space X is perfectly normal and extremally disconnected and constructed using club. Then $W(X)$ is normal and hereditarily extremally disconnected and is not perfectly normal.

The question then arises as to whether normal hereditarily extremally disconnected spaces have the property that every subset is TC^*-embedded, particularly under MA$+\sim$CH.

Finally, the author would like to thank the referee for his many helpful suggestions.

REFERENCES

1. C. E. Aull, *Extendability and expandability* (submitted for publication).
2. R. L. Blair, *Spaces in which special sets are z-embedded*, Canad. J. Math. **28** (1976), 673–690.
3. R. L. Blair and A. W. Hager, *Extensions of zero-sets and of real-valued functions*, Math. Z. **136** (1974), 41–52.

4. R. Engelking, *Cartesian products and dyadic spaces*, Fund. Math. **57** (1965), 287–306.
5. ——, *General topology*, PWN, Warsaw, 1977.
6. L. Gillman and M. Jerison, *Rings of continuous functions*, Van Nostrand, Princeton, N. J., 1966.
7. T. Isiwata, *Mappings and spaces*, Pacific J. Math. **20** (1967),455–480.
8. I. Juhasz, *Cardinal functions in topology*, Math. Centre Tracts, No. 34, Mathematisch Centrum, Amsterdam, 1971.
9. R. Pol, *Short proofs of two theorems of cardinality of topological spaces*, Bull. Acad. Polon. Sci. Sér. Math. **22** (1974), 1245–1249.
10. M. Wage, *Extremally disconnected S-spaces*, Topology Proc. (Conf. Auburn Univ. Auburn, Ala, 1976), Math. Dept., Auburn Univ., Auburn, 1977, pp. 181–186.
11. R. G. Woods, *Absolutes of topological spaces*, Math. Centre Tracts, No. 116, Mathematisch Centrum, Amsterdam, 1979, pp. 323–362.

Department of Mathematics, Virginia Polytechnic Institute and State University, Blacksburg, Virginia 24061