EMBEDDING COSMIC SPACES IN LUSIN SPACES

AMER BEŠLAGIĆ

Abstract. We show that every regular cosmic space can be embedded in a Lusin space. This answers a question posed by J. P. R. Christensen.

In his book [2], J. P. R. Christensen asks the following question: Can a regular cosmic space be embedded in an analytic space?

The purpose of this note is to give a positive answer to that question. The answer was also obtained by Gary Gruenhage and by Calbrix [1].

For undefined terminology we refer the reader to [4]. By regular we mean $T_3$. A $T_0$ space is called cosmic if it is a continuous image of a separable metric space. Michael [7] defined cosmic spaces and proved the following theorem.

Theorem 0. A $T_0$ space is cosmic if and only if it has a countable network.

A network for a space $X$ is a family $\mathcal{N}$ of subsets of $X$ (not necessarily open) such that for every $x \in X$ and open $U$ containing $x$ there is an $N \in \mathcal{N}$ with $x \in N \subseteq U$.

A $T_2$ space $X$ is called analytic if it is a continuous image of a complete separable metric space. A $T_2$ space $X$ is called Lusin if it is a one-to-one continuous image of a complete separable metric space.

A centered system is a family of sets with the finite intersection property, and a centered system on a family $\mathcal{N}$ is a centered system whose members belong to $\mathcal{N}$. A centered system $\mathcal{F}$ of subsets of a space $X$ converges to a point $x$ of $X$ if every neighbourhood of $x$ contains an element of $\mathcal{F}$. Let $\mathcal{N} \subseteq \mathcal{P}(X)$, then for $A \subseteq X$ define $[A]_{\mathcal{N}} = \{x \in X : \text{there is a maximal centered system } \mathcal{F} \text{ on } \mathcal{N} \text{ which contains } A \text{ such that } \mathcal{F} \text{ converges to } x \}$.

A family $\mathcal{N} \subseteq \mathcal{P}(X)$ is convergent if every maximal centered system $\mathcal{F}$ on $\mathcal{N}$ converges to some point $x$ such that for any neighbourhood $U$ of $x$ there is an $A \in \mathcal{F}$ such that $[A]_{\mathcal{N}} \subseteq U$.

Theorem 1. A $T_2$ space $X$ is analytic if and only if it has a countable convergent network.

Proof. Suppose $\mathcal{N} = \{N_n : n \in \omega\}$ is such a network. Without loss of generality we may assume that $\mathcal{N}$ is closed under finite intersections. For $x \in X$ let $\mathcal{F}_x$ denote a maximal centered system on $\mathcal{N}$ such that $\{N \in \mathcal{N} : x \in N\} \subseteq \mathcal{F}_x$. Choose $f_x \in \omega^\omega$ by the rules

$$f_x(n) = n \quad \text{if } N_n \in \mathcal{F}_x,$$

otherwise

$$f_x(n) = \min\{k : N_k \cap N_n = \emptyset \& N_k \in \mathcal{F}_x\}.$$
The closure $M$ of $\{f_x : x \in X\}$ in $ω$ is a complete separable metric space. If $f \in M$, then $\{N_{f(i)} : i \in ω\}$ is a centered system on $N$ and there is a maximal centered system $F_f$ extending it. By assumption, $F_f$ converges to say $x_f \in X$. We claim the map $f \mapsto x_f$ is continuous. Thus $X$ is analytic, for the fact that $X$ is $T_2$ assures us that $x_{f_x} = x$, and thus the map is onto.

To prove the claim, suppose $x_f \in U$ which is open in $X$. There is an $n$ with $x_f \in [N_n]_U \subset U$ and $N_n \in F_f$. If $f(n) = i \neq n$ then there is $x \in X$ with $f(n) = f_x(n)$ so $N_i \cap N_n = \emptyset$, a contradiction. So $\{g \in M : g(n) = n\}$ is an open set in $M$ containing $f$ and we claim that $\{x_g : g(n) = n\} \subset U$, proving that $f \mapsto x_f$ is continuous. But $F_g$ converges to $x_g$ and $N_n \in F_f$ so $x_g \in [N_n]_U$.

Going the other way assume $f : M \to X$ is continuous for some complete separable metric space $M$ and $T_2$ space $X$. For each $x \in X$ choose $x' \in f^{-1}(x)$ and let $M'$ be the closure in $M$ of $\{x' : x \in X\}$. For each $n \in ω$ choose a star finite cover $B_n$ of $M'$ by closed sets of diameter $< 1/2^n$. Let $B = \bigcup_{n \in ω} B_n$. For $B \in B$ let $B' = \{x' : x \in X\}$ and $N = \{f(B') : B \in B\}$.

1°. $N$ is a network for $X$: If $x \in U$ which is open in $X$, choose $B \in B$ with $x' \in f^{-1}(U)$. Then $f(B') \subset U$ so $N$ is a network.

2°. $N$ is convergent: If $F$ is a maximal centered system on $N$ then $\{B : f(B') \in F\}$ is a centered system and there is a (unique) $p \in \bigcap \{B : f(B') \in F\}$ since each $B \in B_i$ has diameter $< 1/2^i$ and meets only finitely many other members of $B_i$ and the metric is complete. We claim that $F$ converges to $f(p)$ and that for any neighbourhood $U$ of $f(p)$ there is an $A \in F$ so that $[A]_N \subset U$. To prove that, choose $B \subset f^{-1}(U)$ with $f(B') \in F$. If $x \in [f(B')]_N$ there is $q \in M$ so that $q \in B$ and $f(q) = x$ so $[f(B')]_N \subset f(B) \subset U$.

The inner characterization of regular analytic spaces is a bit simpler.

**Corollary.** A regular space $X$ is analytic if and only if it has a countable network $N$ such that every maximal centered system on $N$ converges.

**Remark.** Let us mention that Theorem 1 remains true if in the definitions before it one writes infinite instead of maximal. This new version of Theorem 1 seems to be more useful. For example, Hurewicz proved: A metrizable analytic space $Y$ is $σ$-compact if and only if it does not contain a closed copy of the irrationals [5, p. 100], in a rather indirect way. One can use the new version of Theorem 1 to give a short alternative direct proof. In fact, that proof only requires $Y$ to be regular (cf. also [3, Lemma 8.8] for another direct proof of the above).

From Theorem 1 one can easily derive an answer to Christensen’s question, but we can prove a bit more.

Let us call a network $N$ complemented if $X - N$ is the union of members of $N$ for every $N \in N$.

**Theorem 2.** A $T_2$ space $X$ is Lusin if and only if $X$ has a countable complemented network $N$ such that every centered system on $N$ has an intersection in $X$.

**Proof.** Suppose that $f : M \to X$ is continuous and one-to-one for some complete separable metric space $M$ and $T_2$ space $X$. For each $n \in ω$ choose a countable star finite closed cover $B_n$ of $M$ by sets of diameter $< 1/2^n$. Let $B = \bigcup_{n \in ω} B_n$ and $N = \{f(B) : B \in B\}$. Since $f$ is continuous and one-to-one and $B$ is complemented,
$\mathcal{N}$ is a complemented network. If $\mathcal{F}$ is a centered system on $\mathcal{N}$ then $\mathcal{F}' = \{B \in \mathcal{B} : f(B) \in \mathcal{F}\}$ is also centered and there is a $y \in \bigcap \mathcal{F}'$. But $f(y) \in \bigcap \mathcal{F}$. Thus $\mathcal{N}$ is as desired.

Going the other way, assume that $\mathcal{N} = \{N_n : n \in \omega\}$ is a complemented network for $X$ and every centered system on $\mathcal{N}$ has a nonempty intersection. If $x \in X$, define $f_x \in \omega^\omega$ by the rules

$$f_x(n) = n \quad \text{if } x \in N_n \quad \text{and} \quad f_x(n) = \min\{i : \text{N}_i \cap N_n = \emptyset \& x \in N_i\} \quad \text{if } x \notin N_n.$$ 

Since $\mathcal{N}$ is complemented this is possible.

Suppose $M = \{f_x : x \in X\}$ and $f \in \bar{M}$. Then $\{N_{f(n)} : n \in \omega\}$ is a centered system on $\mathcal{N}$ and, by assumption, there is $x \in \bigcap_{n \in \omega} N_{f(n)}$. We claim that $f = f_x$. For suppose $f(n) \neq f_x(n)$ for some $n$ if $j = f_x(n)$, there is $y \in X$ with $f_y(n) = f(n)$ and $f_y(j) = f(j)$. Since $x \in N_j \cap N_{f(n)}$ $j$ and $f(n)$ are minimal in $\{i : N_i \cap N_j = \emptyset\}$ for $x \in N_j$ and $y \in N_{f(j)}$ respectively. Since $x \in N_{f(j)}$, $j < f(j)$ and $y \notin N_j$. Thus $f_z(j) = j$ and $f_y(j) \neq j$; hence $N_{f_z(j)} \cap N_{f(y)} = \emptyset$ but this is impossible since $x \in N_{f_z(j)} \cap N_{f(y)}$.

Hence $f = f_x$, $M$ is closed, and the map $f_x \mapsto x$ of $M$ onto $X$ is one-to-one.

This map is also continuous since if $x \in U$, open in $X$, $x \in N_n \subseteq U$ and $\{f \in M : f(n) = n\}$ is an open set in $M$, containing $f_x$, which is mapped into $U$. Thus $X$ is Lusin.

**Corollary.** If $X$ is a regular cosmic space, $X$ can be embedded in a Lusin space.

**Proof.** Because of the regularity of $X$ we can choose a closed network $\mathcal{N} = \{N_n : n \in \omega\}$ for $X$. Without loss of generality we may assume that $\mathcal{N}$ is closed under finite intersections. Let $X'$ be the space consisting of all maximal centered systems on $\mathcal{N}$. If $U$ is open in $X$ let

$$U' = \{x' \in X' : \exists N \in \mathcal{N} \cap \mathcal{N}(N \subseteq U \& N \in x')\}.$$ 

Every regular cosmic space is normal so sets $U'$ form a base for a $T_2$ topology on $X'$. If $x \in X$, let $x^m = \{N \in \mathcal{N} : x \in N\}$. As $\mathcal{N}$ is a closed network $x^m$ is a maximal centered system on $\mathcal{N}$. It is easy to check that the mapping $f : X \to X'$ given by $f(x) = x^m$ is an embedding.

For $N \in \mathcal{N}$ define $N' = \{x' \in X' : N \in x'\}$, and let $\mathcal{N}' = \{N'_n : n \in \omega\}$. If $\mathcal{F}'$ is a centered system on $\mathcal{N}'$ then $\{N_n : N_n \in \mathcal{F}'\}$ is a centered system on $\mathcal{N}$ so there is $x' \in \bigcap \mathcal{F}'$. If $x' \notin N'_n$ then there is an $N_m \in x'$ so that $N_m \cap N_n = \emptyset$. So $N'_m \cap N'_n = \emptyset$, hence $\mathcal{N}'$ is a complemented network and so $X'$ is a Lusin space.

Let us note that the assumption of regularity of $X$ is necessary, as shown in [6], and also that the Lusin space $X'$ is, in general, unlikely to be regular, so the question whether or not a regular cosmic space can be embedded in a regular analytic space is still open.

In closing, we would like to mention an open question involving cosmic spaces.

If $\omega X$ is hereditarily Lindelöf and hereditarily sequentially separable, is $X$ cosmic? (A set $D \subseteq X$ is sequentially dense in $X$ if for every $x \in X$ there is a sequence of points of $D$ converging to $x$. A space $X$ is sequentially separable if there is a countable sequentially dense $D \subseteq X$.)

Michael gives an example in [8] which shows that under CH the answer is no; Rudin [9] has proved that under CH there is a subset of the real line with the half-open-interval topology which is an example.
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REFERENCES


Department of Mathematics, University of Wisconsin, Madison, Wisconsin 53706