FINITELY BOOLEAN REPRESENTABLE VARIETIES

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Abstract. This paper gives a short, elementary proof of a result of Burris and McKenzie [2] stating that each variety Boolean representable by a finite set of finite algebras is the join of an abelian and a discriminator variety. An example showing that the Boolean product operator \( \Gamma^n \) is not idempotent is included as well.

1. Introduction. The result just mentioned was obtained as a corollary of the authors' description of locally finite decidable varieties with modular congruence lattices, and it was asked in Freese and McKenzie [3] whether there is a "reasonable" proof of this statement. We hope that the one given in the next section is such. In fact, we prove a bit more.

Theorem. Let \( V \) be a variety generated by a finite set \( \mathcal{K} \) of finite algebras with the property that the countable members of \( V \) are in \( \Gamma^n(\mathcal{K}) \). Then \( V \) is the join of an abelian and a discriminator variety (which are independent).

We assume that the reader is familiar with the concept of modular commutator as well as that of Boolean product; these notions together with a complete background of the problem are found in Freese and McKenzie [3].

2. The proof. We fix a variety \( V \) satisfying the conditions of the Theorem. By the results of McKenzie [5] (also mentioned in [3]), we may assume that \( V \) is congruence permutable and each directly indecomposable algebra of \( V \) is finite, and is either simple or abelian. Our reasoning is based on the following concept.

Definition. A subalgebra \( \mathfrak{U} \) of an algebra \( \mathfrak{B} \) is called very skew if \( \mathfrak{U} \) is skew in each direct decomposition of \( \mathfrak{B} \), that is, for nontrivial congruences \( \theta, \psi \) of \( \mathfrak{B} \) with \( \theta \circ \psi = 1 \), \( \theta \cdot \psi = 0 \) we have

\[
(\theta \uparrow \mathfrak{U}) \circ (\psi \uparrow \mathfrak{U}) < 1_{\mathfrak{U}}.
\]

First, we show that a variety is of the desired type iff the powers of its neutral simple algebras have no large very skew subalgebras.

Lemma 1. Let \( V \) be a finitely generated modular variety with the property that each directly indecomposable member of \( V \) is either simple or abelian. Then \( V \) is the join of an abelian and a discriminator variety iff for each neutral simple member \( \mathfrak{B}_0 \) of \( V \) there exists a natural number \( k \) such that the very skew subalgebras of the finite direct powers of \( \mathfrak{B}_0 \) admit at most \( k \) elements.
Before proving this statement, we borrow an observation from [4] (where all modular varieties with complemented principal congruences are described) which will be useful later.

**Lemma 2.** Let the algebra $\mathcal{C}$ (in a modular variety) be a subdirect product of some neutral simple algebras $\mathcal{C}_i$ ($i \in I$), and let $\theta, \psi$ be complements in the congruence lattice of $\mathcal{C}$. Then for some subsets $A$ and $B = I - A$ of $I$, the congruences $\theta$ and $\psi$ are just the kernels of the projections to $A$ and $B$, respectively.

**Sketch of Proof.** Let $\phi_i$ be the kernel of the projection to $\mathcal{C}_i$. Then as $\mathcal{C}_i$ is neutral and simple, each $\phi_i$ is either over $\theta$ or over $\psi$. Now the choice $A = \{i | \phi_i \geq \theta\}$ works by modularity.

Let us now prove Lemma 1. The ‘only if’ part is clear by standard arguments (see [6]), and by [1, Corollary 9.9] it suffices to show that if $\mathcal{A}_0$ is a neutral simple algebra of $\mathcal{V}$, and $\mathcal{A}_0$ is a nonsingleton subalgebra of $\mathcal{A}_0$, then $\mathcal{A}_0$ is neutral and simple.

First, let $\phi$ be a nontrivial congruence of $\mathcal{A}_0$, $\mathcal{V} = "(\mathcal{V}_0)$, and set

$$\mathcal{A} = \{b \in \mathcal{V}_0 | b_i \phi b_j (i, j \in n)\} \subseteq \mathcal{V}$$

(this construction is standard in commutator theory). Now $\mathcal{A}$ is very skew if $\phi \neq 1$ (since the direct decompositions of $\mathcal{V}$ are the obvious ones by Lemma 2), and if $\phi \neq 0$, then $\mathcal{A}$ has at least $2^n$ elements, which is a contradiction.

Secondly, suppose the $\mathcal{A}_0$ is abelian, set $\mathcal{V} = \mathcal{V}_0$ and

$$\mathcal{A} = \{b \in \mathcal{V}_0 | b_0 + \cdots + b_{n-1} = b_n + \cdots + b_{2^n-1}\}.$$ 

where $+$ is an abelian group addition on $\mathcal{A}_0$ compatible with the fundamental operations. This $\mathcal{A}$ is a subalgebra of $\mathcal{V}$ of cardinality $|\mathcal{A}_0|2^{n-1}$. Furthermore, the images of $\mathcal{A}$ and $\mathcal{V}_0$ are the same in each proper factor of $\mathcal{V}$, thus $\mathcal{A}$ is again very skew.

The appropriate sensitive tool for investigating very skew subalgebras seems to be the following.

**Definition.** Let $\mathcal{A} \leq \mathcal{V}$ be algebras. Set

$$\mathcal{V}_{[\mathcal{A}]} = \{u \in \mathcal{V}| \exists a \in \mathcal{A} : \{i | u_i \neq a\} \text{ is finite}\}.$$ 

We denote this element $a$ by $\tilde{u}$, and for $b \in \mathcal{V}$ let $\tilde{b}: \omega \to \mathcal{V}$ be the constant $b$ mapping.

It will turn out that $\mathcal{A}$ “splits nicely” if we make direct decompositions of $\mathcal{V}$ with the aid of a Boolean product representation of $\mathcal{V}_{[\mathcal{A}]}$. More precisely, we show that if $\mathcal{V} = "(\mathcal{V}_0)$ for some neutral simple (finite) $\mathcal{V}_0$, and $\mathcal{A} \leq \mathcal{V}$ is very skew, then $\mathcal{V}_{[\mathcal{A}]}$ has only trivial direct decompositions, and hence if $\mathcal{V}_{[\mathcal{A}]}$ is in $\Gamma^a(\mathcal{H})$, then some element of $\mathcal{H}$ majorates $\mathcal{A}$ in power. Let us fix the algebras $\mathcal{V}_0, \mathcal{A}, \mathcal{V}$ as in the previous sentence.

**Lemma 3.** Suppose that $\mathcal{V}_{[\mathcal{A}]} = \mathcal{C}_1 \times \mathcal{C}_2$. Then either $\mathcal{C}_1$ or $\mathcal{C}_2$ is finite. If $a_1, a_2$ are different elements of $\mathcal{A}$ then the images of $\tilde{a}_1$ and $\tilde{a}_2$ in the “cofinite” component of this direct decomposition are also different.
Proof. We have a subdirect decomposition

\[ \mathfrak{B}_{|\mathfrak{A}|} \leq \prod_{n \times \omega} \mathfrak{B}_0. \]

Let \( \theta, \psi \) be the congruences corresponding to the decomposition \( \mathfrak{B}_{|\mathfrak{A}|} \cong \mathbb{C}_1 \times \mathbb{C}_2 \), and \( A, B \subseteq n \times \omega \) be the subsets given by Lemma 2. It clearly suffices to show that either \( A \) or \( B \) is finite.

For each \( i \in \omega \) we get a direct decomposition of \( \mathfrak{B} \) from that of \( \mathfrak{B}_{|\mathfrak{A}|} \): it is determined by the subsets

\[ A_i = \{ j \in n \mid (j, i) \in A \} \quad \text{and} \quad B_i = \{ j \in n \mid (j, i) \in B \} \]

of \( n \). We prove that disregarding finitely many indices \( i \), this decomposition is trivial. Indeed, otherwise, there is an infinite \( I \subseteq \omega \) such that \( A_i \) and \( B_i \) are the same subsets, say \( A' \) and \( B' \), of \( n \), respectively, for \( i \in I \); and \( A', B' \neq \emptyset \). Let \( \theta', \psi' \) denote the congruences of \( \mathfrak{B} \) corresponding to its direct decomposition determined by \( A' \) and \( B' \). As \( \mathfrak{A} \) is very skew, there exist elements \( a_1, a_2 \) of \( \mathfrak{A} \) such that

\[ (a_1, a_2) \in (\theta' \circ \mathfrak{A}) \circ (\psi' \circ \mathfrak{A}). \]

On the other hand, \( (\tilde{a}_1, \tilde{a}_2) \in \theta \circ \psi \), say \( \tilde{a}_1 \theta \psi \tilde{a}_2 \), and as \( I \) is infinite, we clearly have \( \tilde{a}_1 \theta \tilde{a}_2 \), which is a contradiction.

Suppose now that \( A_i = \emptyset \), as well as \( B_i = \emptyset \), hold infinitely many times. Choose arbitrary elements \( a_1 \neq a_2 \) from \( \mathfrak{A} \). Then with some \( \tilde{a}_1 \theta \psi \tilde{a}_2 \) we clearly have \( a_1 = \tilde{a} = a_2 \), which is a contradiction. Thus, either \( A \) or \( B \) is finite, as desired.

The proof of the Theorem will be complete by showing

Lemma 4. If \( \mathfrak{B}_{|\mathfrak{A}|} \leq \Gamma^a(\mathfrak{K}) \), then there exists a \( \mathfrak{A} \in \mathfrak{K} \) such that \( |\mathfrak{A}| \leq |\Omega| \).

Proof. We have

\[ \mathfrak{B}_{|\mathfrak{A}|} \leq \prod_{i \in I} \Omega_i. \]

If for some \( i \in I \), the \( i \)th components of the elements \( \tilde{a} (a \in \mathfrak{A}) \) are all different, then clearly \( |\mathfrak{A}| \leq |\Omega_i| \). Otherwise, we have (with \( \ll x = y \) being the equalizer of \( x \) and \( y \))

\[ \bigcup_{a_1 \neq a_2 \in \mathfrak{A}} \ll \tilde{a}_1 = \tilde{a}_2 \subseteq I. \]

Thus we obtain a partition of \( I \) into the clopen sets \( A_1, \ldots, A_s \) with the property that each \( A_i \) is covered by some \( \ll \tilde{a}_1 = \tilde{a}_2 \). This partition defines a direct decomposition of \( \mathfrak{B}_{|\mathfrak{A}|} \) which does not satisfy Lemma 3.

3. \( \Gamma^0 \) is not idempotent. If all the elements of \( \mathfrak{K} \) are either affine or simple, then we have a straightforward proof to the Theorem by constructing a term in \( F_v(\omega) \) which is the discriminator in each maximal neutral simple algebra of \( V \); and each variety representable by any \( \mathfrak{K} \) can be represented by such a \( \mathfrak{K} \) by the Theorem and [6]. However, our previous argument shows that we cannot in general assume that \( \mathfrak{K} \) is so nice. Indeed, let \( \mathfrak{B}_0 \) be the alternating group on five letters, \( \mathfrak{A}_0 \) a two-element subgroup of \( \mathfrak{B}_0 \) and let \( \mathfrak{B}, \mathfrak{A} \) be as in the “abelian” construction of the proof of
Lemma 1 for some $n$ with the property that $|\mathfrak{A}| = 2^{2^n-1} > 2^{2^n}$. Then, by Lemma 4, we have

$$\mathfrak{A} | \mathfrak{A} | \in \Gamma^a(\mathfrak{A}, \mathfrak{B}) - \Gamma^a(\mathfrak{A}_0, \mathfrak{B}_0),$$

and since clearly $\mathfrak{A}, \mathfrak{B} \in \Gamma^a(\mathfrak{A}_0, \mathfrak{B}_0)$, this example shows also that the operator $\Gamma^a$ is not idempotent.

REFERENCES


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