PHRAGMÉN-LINDELOF THEOREM IN A COHOMOLOGICAL FORM

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ABSTRACT. The main result of this paper is as follows. Given functions \( \phi_1(e), \ldots, \phi_c(e) \) which are holomorphic in sectors \( S_1, \ldots, S_c \), respectively, where \( S_1 \cup \cdots \cup S_c = \{ e : |\arg e| < \pi/2a, 0 < |e| < \rho \} \) for \( a > 1, \rho > 0 \), set \( \phi_{jk} = \phi_j - \phi_k \) if \( S_j \cap S_k \neq \emptyset \).

Then \( \{ \phi_{jk} \} \) satisfy cocycle conditions \( \phi_{jk} + \phi_{kj} = \phi_{lj} \) whenever \( S_j \cap S_k \neq \emptyset \). In addition to the conditions \( |\phi_1| \leq M_0 \) and \( |\phi_c| \leq M_0 \) on the two rays of the boundary (i.e. \( \arg e = \pi/2a \)), and \( |\phi_1(e)| \leq A \exp(c/|e|) \) in \( S_j \) for some positive numbers \( A \) and \( c, j = 1,2,\ldots,c \).

If the \( \{ \phi_j \} \) satisfy the conditions \( |\phi_j| \leq M_0 \) on \( S_j \cap S_k \neq \emptyset \), then we get \( |\phi_j| \leq M \) on \( S_j \), \( j = 1,2,\ldots,c \) (From the cohomological point of view, we can get global results for \( \phi_j \), once the local data on cocycles is known.)

1. Introduction. Let \( \Omega = \{ e : -\pi/2a < \arg e < \pi/2a, 0 < |e| < \rho \} \) be a sector in the right half complex \( e \)-plane, where \( a > 1 \) and \( \rho \) is a positive number. Let \( f \) be a complex valued function which is continuous on \( \Omega^* = \{ e : -\pi/2a < \arg e < \pi/2a, 0 < |e| < \rho \} \), holomorphic in \( \Omega \), and there are positive constants \( M \) and \( c \) such that \( |f(e)| \leq M \exp(c/|e|) \) for all \( e \in \Omega \). Assume, furthermore, that \( |f(e)| \leq M \) for all \( e \) on the boundary of \( \Omega \). Then, the Phragmén-Lindelöf theorem states that \( |f(e)| \leq M \) for all \( e \) in \( \Omega \) (see [2, p. 282]). In this paper, we shall generalize this theorem in a cohomological form; i.e. given functions \( \phi_1(e), \ldots, \phi_c(e) \) which are holomorphic in sectors \( S_1, \ldots, S_c \), respectively, \( \Omega = S_1 \cup \cdots \cup S_c \), set \( \phi_{jk} = \phi_j - \phi_k \) if \( S_j \cap S_k \neq \emptyset \).

Then \( \{ \phi_{jk} \} \) satisfy cocycle conditions \( \phi_{jk} + \phi_{kj} = \phi_{lj} \) whenever \( S_j \cap S_k \neq \emptyset \). With this property, our theorem can be stated in the following way: If the \( \{ \phi_j \} \) satisfy the conditions \( |\phi_j| \leq M_0 \) on \( S_j \cap S_k \neq \emptyset \) in addition to the conditions \( |\phi_1| \leq M_0 \) and \( |\phi_c| \leq M_0 \) on the two rays of the boundary (i.e. \( \arg e = \pm \pi/2a \)), then we get \( |\phi_j| \leq M \) on \( S_j \), \( j = 1,\ldots,c \) (cf. Theorem 1). From the cohomological point of view, we can get global results for \( \phi_j \) once the local data on cocycles is known. In our theorem, we have chosen those covering sectors \( S_1, \ldots, S_c \) in a nice situation in which \( S_j \cap S_k \neq \emptyset \) only for \( k = j - 1 \) or \( j + 1, j = 2,\ldots,c - 1 \) (cf. (2.1)). The following example will give a prototype of our result.

Let us consider two sectors \( S_j = \{ e : a_j < \arg e < b_j, 0 < |e| < \rho \} \) \( (j = 1,2) \) where \(-\pi/2a = a_1 < a_2 < b_1 < b_2 = \pi/2a \). Note that \( S_1 \cup S_2 = \{ e : -\pi/2a < \arg e < \pi/2a, 0 < |e| < \rho \} \) and \( S_1 \cap S_2 \neq \emptyset \) (cf. Figure 1).
Let $\phi_j$ be functions of $e$ which are, respectively,

1. holomorphic in $S_j$,
2. continuous on $S_j^* = \{e: a_j \leq \arg e \leq b_j, 0 < |e| \leq \rho\}$, and
3. $|\phi_j(e)| \leq A \exp(c/|e|)$ in $S_j$ for some positive numbers $c$ and $A$.

Assume that $|\phi_2(e) - \phi_1(e)| \leq M_0$ in $S_1 \cap S_2$, $|\phi_i(e)| \leq M_0$ on the line segment $\arg e = -\pi/2\alpha$, $0 < |e| < \rho$, and $|\phi_2(e)| \leq M_0$ on the line segment $\arg e = \pi/2\alpha$, $0 < |e| < \rho$, for some $M_0$. Then we claim that $|\phi_j(e)| \leq M$ in $S_j$ ($j = 1, 2$), respectively, for some positive number $M$ (cf. Theorem 1).

The difficulty in proving this result is due to the fact that $\phi_1$ and $\phi_2$ are two different functions in $S_1 \cap S_2$, and hence a straightforward application of the Phragmén-Lindelöf theorem does not work.

The proof of this result, may be obtained by defining an auxiliary function (as in the proof of the Phragmén-Lindelöf theorem) together with Theorem 2 (cf. §2). Previously, Y. Sibuya [3] proved a result similar to Theorem 2, in the case when $S = S_1 \cup \cdots \cup S_p$ is a disk. However, he did not show that $M$ was actually independent of $c_1$ as $c_1$ tends to zero. In the proof of Theorem 1, we shall let $c_1$ of Theorem 2 tend to zero (as in the proof of the Phragmén-Lindelöf theorem). Since $M$ in Theorem 2 is independent of $c_1$ as $c_1$ tends to zero, such a process will work. Notice also that if $\bigcup_{j=1}^p S_j$ is a disc, there are no boundary rays. In this paper, $\bigcup_{j=1}^p S_j$ being a sector, the analysis near the boundary rays becomes another additional difficulty (cf. Proof of Theorem 2 in §3).

Given functions $\delta_1(e), \ldots, \delta_p(e)$ which are holomorphic in sectors $S_1, \ldots, S_p$, respectively, $S_1 \cup \cdots \cup S_p = \{e: |\arg e| < \pi/2\alpha, 0 < |e| < \rho\}$, satisfying certain reasonable assumptions and having relatively poor estimates on the two rays of the
boundary (i.e. \( \arg \epsilon = \pm \pi/2\alpha \)) which are assumed to be close to the imaginary axis, we can substantially improve such estimates along the real axis utilizing Theorem 3 (cf. Theorem 3 in §2). The usefulness of our results is due to the fact that, in general, it is very difficult to get a global result all at once. In some cases where the Phragmén-Lindelöf theorem cannot be applied directly, we still can obtain a global result by putting suitable local results together through an application of our theorem (cf. e.g. [1 and 3]).

2. The statement of theorems.

**Theorem 1.** Let \( S_j = \{ \epsilon: a_j < \arg \epsilon < b_j, \ 0 < |\epsilon| < \rho \} \ (j = 1, \ldots, v) \) be sectors in the right half complex \( \epsilon \)-plane where \( \rho > 0 \):

\[
-\pi/2\alpha = a_1 < a_2 < b_1 < a_3 < b_2 < a_4 < b_3 < \cdots < a_v < b_{v-1} < b_v = \pi/2\alpha,
\]

\( \alpha > 1 \).

Let \( \phi_1, \ldots, \phi_v(\epsilon) \) be functions of \( \epsilon \). Assume that

1. \( \phi_j(\epsilon) \) is holomorphic in \( S_j \) and continuous on \( S_j^* = \{ \epsilon: a_j \leq \arg \epsilon \leq b_j, \ 0 < |\epsilon| \leq \rho \} \),
2. \( |\phi_j(\epsilon)| \leq A \exp(c/|\epsilon|) \) in \( S_j \), for some positive numbers \( A \) and \( c \),
3. \( |\phi_j(x) - \phi_j(\epsilon)| < M_0 \exp(-c/|\epsilon|^N) \) in \( S_j \cap S_{j+1} \) \( |\phi_j(\epsilon)| < M_0 \) on the line segment \( \arg \epsilon = -\pi/2\alpha, \ 0 < |\epsilon| < \rho \), and \( |\phi_j(\epsilon)| < M_0 \) on the line segment \( \arg \epsilon = \pi/2\alpha, \ 0 < |\epsilon| < \rho \), for some positive number \( M_0 \). Then, there exists a positive number \( M \) such that

\[
|\phi_j(\epsilon)| < M \text{ in } S_j, \ j = 1, 2, \ldots, v.
\]

**Remark.** The inequalities (2.1) mean that \( S_j \cap S_k \neq \emptyset \) only for \( k = j - 1 \) or \( j + 1, j = 2, 3, \ldots, v - 1 \).

**Theorem 2.** Let \( S_j = \{ \epsilon: a_j < \arg \epsilon < b_j, \ 0 < |\epsilon| < \rho \} \ (j = 1, 2, \ldots, v) \) be sectors in the complex \( \epsilon \)-plane, where \( \rho > 0 \):

\[
-\pi < a_1 < a_2 < b_1 < a_3 < b_2 < a_4 < b_3 < \cdots < a_v < b_{v-1} < b_v < \pi.
\]

Let \( \delta_1(\epsilon), \ldots, \delta_v(\epsilon) \) be functions of \( \epsilon \). Assume that

1. \( \delta_j(\epsilon) \) is holomorphic in \( S_j \), continuous on \( S_j^* \),
2. \( \delta_j(\epsilon) \) is asymptotically zero as \( \epsilon \) tends to zero in \( S_j \), i.e. \( |\delta_j(\epsilon)| \leq K_N |\epsilon|^N \) \( (N = 0, 1, 2, \ldots) \) in \( S_j \) for some positive numbers \( K_N \),
3. \( |\delta_{j+1}(\epsilon) - \delta_j(\epsilon)| < M_0 \exp(-c_1/|\epsilon|^N) \) in \( S_j \cap S_{j+1} \), \( |\delta_j(\epsilon)| < M_0 \exp(-c_1/|\epsilon|^N) \) on the line segment \( \arg \epsilon = a_1, \ 0 < |\epsilon| < \rho \), and \( |\delta_j(\epsilon)| < M_0 \exp(-c_1/|\epsilon|^N) \) on the line segment \( \arg \epsilon = b_\nu, \ 0 < |\epsilon| < \rho \), for some positive numbers \( c_1, M_0 \) and \( N \). Then, there exists a positive number \( M \) which is independent of \( c_1 \) as \( c_1 \) tends to zero such that

\[
|\delta_j(\epsilon)| < M \exp(-c_1/|\epsilon|^N) \text{ in } S_j, \ j = 1, 2, \ldots, v.
\]

**Theorem 3.** Let \( S_j = \{ \epsilon: a_j < \arg \epsilon < b_j, \ 0 < |\epsilon| < \rho \} \) be sectors in the right half complex \( \epsilon \)-plane, where \( \rho > 0 \):

\[
-\pi/2\alpha = a_1 < a_2 < b_1 < a_3 < b_2 < a_4 < b_3 < \cdots < a_v < b_{v-1} < b_v = \pi/2\alpha,
\]

\( \alpha > 1 \).
Let \( \delta_1(\epsilon), \ldots, \delta_v(\epsilon) \) be functions of \( \epsilon \). Assume that

1. \( \delta_j(\epsilon) \) is holomorphic in \( S_j \), continuous on \( S_j^* \),
2. \( \delta_j(\epsilon) \) is asymptotically zero, as \( \epsilon \) tends to zero in \( S_j \),
3. \( |\delta_{j+1}(\epsilon) - \delta_j(\epsilon)| \leq M_0 \exp\{-\mu \Re(1/\epsilon)\} \) in \( S_j \cap S_{j+1} \),

\[ |\delta_j(\epsilon)| \leq M_0 \exp\{-\mu \Re(1/\epsilon)\} \]
on the line segment \( \arg \epsilon = -\pi/2\alpha, 0 < |\epsilon| < \rho \), and

\[ |\delta_v(\epsilon)| \leq M_0 \exp\{-\mu \Re(1/\epsilon)\} \]
on the line segment \( \arg \epsilon = \pi/2\alpha, 0 < |\epsilon| < \rho \), for some positive numbers \( \mu \) and \( M_0 \).

Then, there exists a positive number \( M \) such that \( |\delta_j(\epsilon)| \leq M \exp\{-\mu \Re(1/\epsilon)\} \) in \( S_j \), \( j = 1, 2, \ldots, v \).

3. Proof of the theorems.

(1) **Proof of Theorem 2.** We denote by \( V_j \) the intersections \( S_j \cap S_{j+1}, j = 1, 2, \ldots, v - 1 \), respectively, and consider an open sector

\[ S = \{ \epsilon: a_i < \arg \epsilon < b_i, 0 < |\epsilon| < \rho_0 \}, \text{ where } 0 < \rho_0 < \rho. \]

We choose \( v - 1 \) line segments \( l_1, l_2, \ldots, l_{v-1} \) such that \( l_j \subset V_j \), i.e. \( l_j: \epsilon = te^{i\alpha_j}, (0 < t < \rho_0, \text{ for some } \alpha_j \text{ in } (a_{j+1}, b_j)) \). These \( v - 1 \) line segments divide the open sector \( S \) (cf. (3.1)) into \( v \) open sectors \( S_1, S_2, \ldots, S_v \) (cf. Figure 2).
Let \( \gamma_j (j = 1, 2, \ldots, \nu) \) be the circular arcs which are defined by \( e = \rho_0 e^{i \xi} (\alpha_{j-1} \leq \xi \leq \alpha_j) \), respectively, where \( \alpha_0 = a_1, \alpha_v = b_v \). Set, \( l_0: \ e = te^{i \alpha_1} \) \( (0 < t < \rho_0) \), \( l_v: \ e = te^{i \beta} \) \( (0 < t < \rho_0) \). Then \( \gamma_1 + \gamma_2 + \cdots + \gamma_\nu = C = \{ e: |e| = \rho_\nu, a_1 \leq \arg e \leq b_\nu \} \). The boundaries of \( \hat{S}_1, \hat{S}_2, \ldots, \hat{S}_\nu \) are, respectively, \( l_{j-1} + \gamma_j - l_j, j = 1, 2, \ldots, \nu \).

For \( e \in \hat{S}_1 \cup \hat{S}_2 \cup \cdots \cup \hat{S}_\nu \), set \( \delta(e) = \delta_j(e) \) if \( e \in \hat{S}_j \). Since

\[
\frac{1}{2\pi i} \int_{l_{j-1} + \gamma_j - l_j} \frac{\delta_j(e)}{e - \zeta} \, d\zeta = \begin{cases} \delta_j(e), & e \in \hat{S}_j, \\ 0, & e \notin \hat{S}_j, \end{cases}
\]

we have

\[
\delta(e) = \frac{1}{2\pi i} \sum_{j=1}^\nu \int_{l_{j-1} + \gamma_j - l_j} \frac{\delta_j(e)}{e - \zeta} \, d\zeta \quad \text{in} \quad \hat{S}_1 \cup \hat{S}_2 \cup \cdots \cup \hat{S}_\nu.
\]

Utilizing \( 1/(\xi - e) = \sum_{m=0}^N \xi^{-(m+1)} e^m + \xi^{N+1}/\xi^{N+1}(\xi - e) \), we derive

\[
\delta(e) = \frac{1}{2\pi i} \sum_{m=0}^N \left\{ \sum_{j=1}^\nu \int_{l_{j-1} + \gamma_j - l_j} \xi^{-(m+1)} \delta_j(e) \, d\xi \right\} e^m \\
+ \left\{ \frac{1}{2\pi i} \sum_{j=1}^\nu \int_{l_{j-1} + \gamma_j - l_j} \frac{\delta_j(e)}{\xi^{N+1}(\xi - e)} \, d\xi \right\} e^{N+1}.
\]

Since \( \delta(e) \) is asymptotically zero as \( e \) tends to zero in \( \hat{S}_1 \cup \hat{S}_2 \cup \cdots \cup \hat{S}_\nu \), the first term of (3.2) must be zero. Hence

\[
\delta(e) = \left\{ \frac{1}{2\pi i} \sum_{j=1}^\nu \int_{l_{j-1} + \gamma_j - l_j} \frac{\delta_j(e)}{\xi^{N+1}(\xi - e)} \, d\xi \right\} e^{N+1}.
\]

Therefore, we arrive at the following formula:

\[
\delta(e) = \frac{1}{2\pi i} \left\{ \int_{l_0} \frac{\delta_1(e)}{\xi^N(\xi - e)} \, d\xi - \int_{l_v} \frac{\delta_\nu(e)}{\xi^N(\xi - e)} \, d\xi \\
+ \sum_{j=1}^{\nu-1} \int_{l_j} \frac{\sigma_j(e)}{\xi^N(\xi - e)} \, d\xi + \int_{c} \frac{\delta(e)}{\xi^N(\xi - e)} \, d\xi \right\} e^N
\]

for \( e \in \hat{S}_1 \cup \hat{S}_2 \cup \cdots \cup \hat{S}_\nu \) and \( N = 1, 2, \ldots \) where \( \sigma_j = \delta_{j+1} - \delta_j \). Construct \( \nu \) open sectors \( \hat{S}_1, \hat{S}_2, \ldots, \hat{S}_\nu \) such that the boundaries of \( \hat{S}_j \) are \( \Omega_j \cup T_j \cup \Omega' \), where

\[
\Omega_j = \{ e: e = |e| e^{i(\alpha_{j-1} + \theta)}, |e| \leq \rho_1 \},
\]

\[
\Omega' = \{ e: e = |e| e^{i(\alpha_j - \theta)}, |e| \leq \rho_1 \},
\]

\[
T_j = \{ e: e = \rho_1 e^{i\theta}, \alpha_{j-1} + \theta \leq \beta \leq \alpha_j - \theta \}, \quad j = 1, 2, \ldots, \nu,
\]

\( 0 < \rho_1 < \rho_0 \) and \( \theta \) is a small positive number such that \( \lambda \theta < \pi \) (cf. Figure 3).
Then, \( \hat{S}_j \subseteq S_j, j = 1, 2, \ldots, v. \)

For \( \varepsilon \in \hat{S}_1 \cup \hat{S}_2 \cup \cdots \cup \hat{S}_v, \)

\[
\left| \int_{\xi} \frac{\delta(\xi)}{\xi^N(\xi - \varepsilon)} d\xi \right| \leq \int_{\xi} \frac{|\delta(\xi)|}{|\xi^N| |\xi - \varepsilon|} |d\xi| \leq \int_{\xi} \frac{K_\delta}{\rho_0^N (\rho_0 - \rho_1)} |d\xi|
\]

\[
= \frac{K_\delta \rho_0}{\rho_0^N (\rho_0 - \rho_1)} (b_v - a_1) = \frac{B_\delta}{\rho_0^{N-1}},
\]

where \( K_\delta = \max_{\xi \in C} |\delta(\xi)|, B_\delta = K_\delta (b_v - a_1)/(\rho_0 - \rho_1). \)

\[
\left| \int_{t_0}^{\rho_0} \frac{\delta_1(t_0)}{\xi^N(\xi - \varepsilon)} d\xi \right| \leq \int_{t_0}^{\rho_0} \frac{\delta_1(t_0)}{t^N e^{t_0 a_1} (te^{a_1} - \varepsilon)} e^{i a_1} dt 
\]

\[
\leq \frac{M_0}{\sin \theta} \int_{t_0}^{\rho_0} t^{-N-1} \exp \left(-c_1/t^\lambda \right) dt 
\]

\[
= \frac{M_0}{\lambda \sin \theta} \int_{t_0}^{+\infty} t^{(N/\lambda) - 1} \exp(-c_1 \tau) d\tau 
\]

\[
= \frac{M_0}{\lambda \sin \theta} C_1^{-(N/\lambda)}(N/\lambda).
\]

Similarly,

\[
\left| \int_{t_0}^{\rho_0} \frac{\delta_0(\xi)}{\xi^N(\xi - \varepsilon)} d\xi \right| < \frac{M_0}{\lambda \sin \theta} C_1^{-(N/\lambda)}(N/\lambda),
\]
and
\[ \left| \int_{l_j} \frac{\sigma_j(\xi)}{\xi^N(\xi - \varepsilon)} \, d\xi \right| < \frac{M_0}{\lambda \sin \theta} \frac{C_j^{-\lambda/N}}{c_{1,\lambda}} \Gamma(\lambda/N), \quad j = 1, 2, \ldots, \nu - 1. \]

Since \( \Gamma(N/\lambda) \leq M_1(N/\lambda)^{(N/\lambda)} e^{-(N/\lambda)} \) for some \( M_1 > 0 \), we have
\[ \left| \int_{l_j} \frac{\delta_j(\xi)}{\xi^N(\xi - \varepsilon)} \, d\xi \right| < \frac{M_0 M_1}{\lambda \sin \theta} \left( \frac{N}{c_{1,\lambda}} \right)^{(N/\lambda)} e^{-(N/\lambda)}, \]
and
\[ \left| \int_{l_j} \frac{\sigma_j(\xi)}{\xi^N(\xi - \varepsilon)} \, d\xi \right| < \frac{M_0 M_1}{\lambda \sin \theta} \left( \frac{N}{c_{1,\lambda}} \right)^{(N/\lambda)} e^{-(N/\lambda)}. \]

for \( \varepsilon \in \tilde{S}_1 \cup \cdots \cup \tilde{S}_\nu, j = 1, 2, \ldots, \nu - 1. \) Then,
\[ |\delta(\varepsilon)| \leq \frac{1}{2\pi} \left( \frac{N}{\lambda \sin \theta} \right)^{(N/\lambda)} e^{-(N/\lambda)} + \frac{B_\delta}{\rho_0^{-1}} \right| \varepsilon \right|^N \]
\[ = \frac{M_2}{2\pi} \left[ 1 + \frac{B_\delta \rho_0}{M_2} \left( \frac{c_{1,\lambda} \varepsilon}{N \rho_0} \right)^{(N/\lambda)} \right] \left( \frac{N}{\varepsilon^\lambda} \right)^{(N/\lambda)} \]
where \( M_2 = (\nu + 1)M_0 M_1/\lambda \sin \theta > 0. \)

Since \( \left( 1 + (B_\delta \rho_0/M_2)(c_{1,\lambda} \varepsilon /N \rho_0)^{(N/\lambda)} \right) \to 1 \) as \( c_1 \to 0, \) we have
\[ |\delta(\varepsilon)| \leq M_3(\delta) \left( \frac{N}{\varepsilon^\lambda} \right)^{(N/\lambda)} e^{-(N/\lambda)} \]
for \( \varepsilon \in \tilde{S}_1 \cup \tilde{S}_2 \cup \cdots \cup \tilde{S}_\nu, \) where \( M_3(\delta) \) is a constant which is independent of \( c_1 \) as \( c_1 \) tends to zero.

For a given \( \varepsilon, \) choose \( N \) so that \( N/\lambda < c_1/|\varepsilon|^\lambda \leq (N + 1)/\lambda. \) We have
\[ \left( 3.4 \right) \quad \frac{N}{|\varepsilon|^\lambda} < c_1 \quad \text{and} \quad \frac{c_1}{|\varepsilon|^\lambda} \geq -\frac{N + 1}{\lambda}. \]

Then, it follows from (3.3) and (3.4) that we have
\[ |\delta(\varepsilon)| \leq M_3(\delta) e^{1/\lambda} \exp \left( -c_1/|\varepsilon|^\lambda \right) \leq M_4 \exp \left( -c_1/|\varepsilon|^\lambda \right), \]
where \( M_4 = \max \{ M_0, M_3(\delta) e^{1/\lambda} \} \) which is independent of \( c_1 \) as \( c_1 \) tends to zero.

Choosing \( l_1, l_2, \ldots, l_{\nu - 1} \) in various ways, we can prove that
\[ |\delta_j(\varepsilon)| \leq M_4 \exp \left( -c_1/|\varepsilon|^\lambda \right) \quad \text{in} \quad S_j', j = 2, \ldots, \nu - 1, \]

where
\[ S'_j = \{ \varepsilon: a_j < \arg \varepsilon < b_j, 0 < |\varepsilon| < \rho_j \}, \quad j = 1, 2, \ldots, \nu. \]

Since \( l_0 \) and \( l_\nu \) are boundaries of \( S \), they cannot be moved. To obtain a similar estimate in \( S'_j \), set \( c_2 = c_1 / \cos(\lambda \theta / 2) \), then
\[ |\delta_i(e^{i(\alpha_1 + \theta / 2)})| \leq M_4 \exp\left(-c_2 \cos(\lambda \theta / 2) / |\varepsilon|^\lambda\right) \]
on
\[ \{ \varepsilon: |\varepsilon| = |e| e^{i\theta / 2}, 0 < |\varepsilon| < \rho_1 \} \cup \{ \varepsilon: |\varepsilon| = |e| e^{-i\theta / 2}, 0 < |\varepsilon| < \rho_1 \}. \]

Let \( h(\varepsilon) = \exp\{c_2 / |\varepsilon|^\lambda\} \) \((\varepsilon \in \{ \varepsilon: 0 < |\varepsilon| < \rho_1 \})\). For \( \arg \varepsilon = \theta / 2, \arg \varepsilon = -\theta / 2, \) we have
\[ |\delta_i(e^{i(\alpha_1 + \theta / 2)}) h(\varepsilon) | \leq M_4 \exp\left\{ -c_2 \cos(\lambda \theta / 2) / |\varepsilon|^\lambda \right\} \cdot \exp\left\{ c_2 \cos(\lambda \theta / 2) / |\varepsilon|^\lambda \right\} = M_4 \]
and for \( \varepsilon = |\varepsilon| e^{i\eta}, -\theta / 2 < \eta < \theta / 2, \) we have
\[ |\delta_i(e^{i(\alpha_1 + \theta / 2)}) h(\varepsilon) | \leq E \exp\left\{ c_2 \cos(\lambda \eta) / |\varepsilon|^\lambda \right\} \leq E \exp\left\{ c_2 / |\varepsilon|^\lambda \right\}, \]
where \( E \) is a bound for \( \delta_i(e^{i(\alpha_1 + \theta / 2)}) \) on \( \{ \varepsilon: -\theta / 2 < \arg \varepsilon < \theta / 2, 0 < |\varepsilon| < \rho_1 \} \). Then, by the Phragmén-Lindelöf theorem, we have
\[ |\delta_i(e^{i(\alpha_1 + \theta / 2)}) h(\varepsilon) | \leq M_4 \exp\left\{ -c_2 \cos(\lambda \theta / 2) / |\varepsilon|^\lambda \right\} \]
i.e. \( |\delta_i(e^{i(\alpha_1 + \theta / 2)})| \leq M_4 \exp\{-c_1 / |\varepsilon|^\lambda\} \) for
\[ \varepsilon = |\varepsilon| e^{i\eta} \in \{ \varepsilon: -\theta / 2 < \arg \varepsilon < \theta / 2, 0 < |\varepsilon| < \rho_1 \}. \]
Thus, \( |\delta_i(\varepsilon)| \leq M_4 \exp\{-c_1 / |\varepsilon|^\lambda\} \) in \( S'_j \); similarly, \( |\delta_j(\varepsilon)| \leq M_4 \exp\{-c_1 / |\varepsilon|^\lambda\} \) in \( S'_j \).
Choosing \( M \) as the maximal value of \( M_4 \) and a bound of \( |\delta_i(\varepsilon)| \exp\{c_1 / \rho_1^\lambda\} \) on \( \{ \varepsilon: a_j < \arg \varepsilon < b_j, \rho_1 < |\varepsilon| < \rho \}, j = 1, 2, \ldots, \nu, \) which is independent of \( c_1 \) as \( c_1 \) tends to zero, we can obtain \( |\delta_j(\varepsilon)| \leq M_4 \exp\{-|\varepsilon|^\lambda\} \) in \( S_j \), \( j = 1, 2, \ldots, \nu. \)

(2) **Proof of Theorem 1.** For each \( \eta > 0 \) we define an auxiliary function
\[ h_\eta(\varepsilon) = \exp\{-\eta / |\varepsilon|^\lambda\} \quad \left( \varepsilon \in \{ \varepsilon: -\pi / 2\alpha < \arg \varepsilon < \pi / 2\alpha, 0 < |\varepsilon| < \rho \} \right) \]
where \( 1 < \lambda < \alpha \). Set \( \delta_j(\varepsilon) = \phi_j(\varepsilon) h_\eta(\varepsilon) \) which depends on \( \eta, j = 1, 2, \ldots, \nu. \) Then, for \( \varepsilon \in l_0, \)
\[ |\delta_i(\varepsilon)| = |\phi_i(\varepsilon) h_\eta(\varepsilon)| \leq M_0 \left| \exp\left\{ -\eta / |\varepsilon|^\lambda e^{-\pi \lambda i / 2\alpha} \right\} \right| \]
\[ = M_0 \exp\left\{ -\eta \cos(\lambda \pi / 2\alpha) / |\varepsilon|^\lambda \right\}; \]
for \( \varepsilon \in l_\nu, \)
\[ |\delta_i(\varepsilon)| = |\phi_i(\varepsilon) h_\eta(\varepsilon)| \leq M_0 \left| \exp\left\{ -\eta / |\varepsilon|^\lambda e^{\pi \lambda i / 2\alpha} \right\} \right| \]
\[ = M_0 \exp\left\{ -\eta \cos(\lambda \pi / 2\alpha) / |\varepsilon|^\lambda \right\}; \]
and for \( \varepsilon = |\varepsilon| e^{i\beta_j} \in S_j \cap S_{j+1}, \)
\[ |\delta_{j+1}(\varepsilon) - \delta_j(\varepsilon)| = |\phi_{j+1}(\varepsilon) - \phi_j(\varepsilon)| \left| h_\eta(\varepsilon) \right| \leq M_0 \exp\left\{ -\eta \cos(\lambda |\beta_j|) / |\varepsilon|^\lambda \right\}. \]
Since $0 < \lambda |\beta_j| < \lambda \pi/2\alpha$, we have $\cos(\lambda |\beta_j|) > \cos(\lambda \pi/2\alpha)$. Then we derive

$$|\delta_{j+1}(\epsilon) - \delta_j(\epsilon)| \leq M_0 \exp\left\{-\eta \cos(\lambda \pi/2\alpha)/|\epsilon|^\lambda \right\} \quad \text{in } S_j \cap S_{j+1},$$

and $\delta_j(\epsilon)$ is asymptotically zero as $\epsilon$ tends to zero in $S_j$. Set $c_1 = \eta \cos(\lambda \pi/2\alpha)$. Then by Theorem 2, there exists a positive number $H$ which is independent of $c_1$ as $c_1$ tends to zero ($c_1 \to 0$, as $\eta \to 0$) such that

$$|\delta_j(\epsilon)| = \left|\phi_j(\epsilon) h_j(\epsilon)\right| \leq M \exp\left\{-\eta \cos(\lambda \pi/2\alpha)/|\epsilon|^\lambda \right\} \quad \text{in } S_j, \quad j = 1, 2, \ldots, \nu.$$

As $\eta \to 0$, $h_j(\epsilon) \to 1$ for every $\epsilon$; so we conclude that $|\phi_j(\epsilon)| \leq M$ in $S_j$, $j = 1, 2, \ldots, \nu$.

(3) Proof of Theorem 3. Let $h(\epsilon) = \exp\{\mu/\epsilon\}$, $\epsilon \in S = \{\epsilon: -\pi/2\alpha \leq \arg \epsilon \leq \pi/2\alpha, 0 < |\epsilon| \leq \rho\}$, and set $\phi_j(\epsilon) = \delta_j(\epsilon) h(\epsilon)$, $j = 1, 2, \ldots, \nu$. Then, for $\epsilon = |\epsilon| e^{\pi i/2\alpha}$,

$$|\phi_1(\epsilon)| = |\delta_1(\epsilon)| |h(\epsilon)| \leq M_0 \exp\{-\mu \text{Re}(1/\epsilon)\} \exp\{\mu \text{Re}(1/\epsilon)\} = M_0;$$

similarly, for $\epsilon = |\epsilon| e^{\pi i/2\alpha}$, $|\phi_j(\epsilon)| \leq M_0$, for $\epsilon = |\epsilon| e^{i\beta_j} \in S_j \cap S_{j+1}$, $|\phi_{j+1}(\epsilon) - \phi_j(\epsilon)| \leq M_0$; and for $\epsilon = |\epsilon| e^{i\beta_j} \in S_j$,

$$|\phi_j(\epsilon)| = |\delta_j(\epsilon)| |h(\epsilon)| \leq A \exp\left\{\mu \cos \xi_j/|\epsilon|\right\} \leq A \exp\{\mu/|\epsilon|\}$$

where $A > 0$ is a bound for $\delta_j$ in $S_j$. By Theorem 1, we have $|\phi_j(\epsilon)| \leq M$ in $S_j$ ($j = 1, 2, \ldots, \nu$), i.e. $|\delta_j(\epsilon)| \leq M \exp\{-\mu \text{Re}(1/\epsilon)\}$ in $S_j$, $j = 1, 2, \ldots, \nu$.

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