PHRAGMÉN-LINDELOF THEOREM
IN A COHOMOLOGICAL FORM

CHING-HER LIN1

Abstract. The main result of this paper is as follows. Given functions \( \phi_1(\epsilon), \ldots, \phi_r(\epsilon) \) which are holomorphic in sectors \( S_1, \ldots, S_r \), respectively, where \( S_1 \cup \cdots \cup S_r = \{ \epsilon: |\arg \epsilon| < \pi/2\alpha, 0 < |\epsilon| < \rho \} \) for \( \alpha > 1, \rho > 0 \), set \( \phi_{jk} = \phi_j - \phi_k \) if \( S_j \cap S_k \neq \emptyset \).

Then \( \{ \phi_{jk} \} \) satisfy cocycle conditions \( \phi_{jk} + \phi_{kj} = \phi_{jl} \) whenever \( S_j \cap S_k \neq \emptyset \).

In addition to the conditions \( |\phi_j| \leq M_0 \) and \( |\phi_k| \leq M_0 \) on the two rays of the boundary (i.e. \( \arg \epsilon = \pm \pi/2\alpha \)), and \( |\phi_j(\epsilon)| \leq A \exp(c/|\epsilon|) \) in \( S_j \) for some positive numbers \( A \) and \( c \), \( j = 1, 2, \ldots, r \), if the \( \{ \phi_j \} \) satisfy the conditions \( |\phi_j| \leq M_0 \) on \( S_j \cap S_k \neq \emptyset \), then we get \( |\phi_j| \leq M \) on \( S_j, j = 1, 2, \ldots, r \). (From the cohomological point of view, we can get global results for \( \phi_j \), once the local data on cocycles is known.)

1. Introduction. Let \( \Omega = \{ \epsilon: -\pi/2\alpha < \arg \epsilon < \pi/2\alpha, 0 < |\epsilon| < \rho \} \) be a sector in the right half complex \( \epsilon \)-plane, where \( \alpha > 1 \) and \( \rho \) is a positive number. Let \( f \) be a complex valued function which is continuous on \( \Omega^* = \{ \epsilon: -\pi/2\alpha < \arg \epsilon < \pi/2\alpha, 0 < |\epsilon| < \rho \} \), holomorphic in \( \Omega \), and there are positive constants \( M \) and \( c \) such that \( |f(\epsilon)| \leq M \exp(c/|\epsilon|) \) for all \( \epsilon \in \Omega \). Assume, furthermore, that \( |f(\epsilon)| \leq M \) for all \( \epsilon \) on the boundary of \( \Omega \). Then, the Phragmén-Lindelöf theorem states that \( |f(\epsilon)| \leq M \) for all \( \epsilon \) in \( \Omega \) (see [2, p. 282]). In this paper, we shall generalize this theorem in a cohomological form; i.e. given functions \( \phi_1(\epsilon), \ldots, \phi_r(\epsilon) \) which are holomorphic in sectors \( S_1, \ldots, S_r \), respectively, \( \Omega = S_1 \cup \cdots \cup S_r \), set \( \phi_{jk} = \phi_j - \phi_k \) if \( S_j \cap S_k \neq \emptyset \).

Then \( \{ \phi_{jk} \} \) satisfy cocycle conditions \( \phi_{jk} + \phi_{kj} = \phi_{jl} \) whenever \( S_j \cap S_k \neq \emptyset \).

With this property, our theorem can be stated in the following way: If the \( \{ \phi_j \} \) satisfy the conditions \( |\phi_{jk}| \leq M_0 \) on \( S_j \cap S_k \neq \emptyset \) in addition to the conditions \( |\phi_j| \leq M_0 \) and \( |\phi_k| \leq M_0 \) on the two rays of the boundary (i.e. \( \arg \epsilon = \pm \pi/2\alpha \)), then we get \( |\phi_j| \leq M \) on \( S_j, j = 1, 2, \ldots, r \) (cf. Theorem 1). From the cohomological point of view, we can get global results for \( \phi_j \) once the local data on cocycles is known. In our theorem, we have chosen those covering sectors \( S_1, \ldots, S_r \) in a nice situation in which \( S_j \cap S_k \neq \emptyset \) only for \( k = j - 1 \) or \( j + 1, j = 2, \ldots, r - 1 \) (cf. (2.1)). The following example will give a prototype of our result.

Let us consider two sectors \( S_j = \{ \epsilon: a_{j-1} < \arg \epsilon < b_j, 0 < |\epsilon| < \rho \} \) (\( j = 1, 2 \)) where \( -\pi/2\alpha = a_1 < a_2 < b_1 < b_2 = \pi/2\alpha \). Note that \( S_1 \cup S_2 = \{ \epsilon: -\pi/2\alpha < \arg \epsilon < \pi/2\alpha, 0 < |\epsilon| < \rho \} \) and \( S_1 \cap S_2 \neq \emptyset \) (cf. Figure 1).

1This work was partially sponsored by the National Science Foundation under grant MCS 79-01998.

This work was also a part of the author's dissertation for a Ph.D of Mathematics at the University of Minnesota, written under the supervision of Professor Sibuya.

©1983 American Mathematical Society
0002-9939/83 $1.00 + $.25 per page

License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use
Let $\phi_j$ be functions of $\epsilon$ which are, respectively,
(1) holomorphic in $S_j$,
(2) continuous on $S_j^* = \{ \epsilon: a_j \leq \arg \epsilon \leq b_j, 0 < |\epsilon| \leq \rho \}$, and
(3) $|\phi_j(\epsilon)| \leq A \exp(c/|\epsilon|)$ in $S_j$ for some positive numbers $c$ and $A$.

Assume that $|\phi_2(\epsilon) - \phi_1(\epsilon)| \leq M_0$ in $S_1 \cap S_2$, $|\phi_1(\epsilon)| \leq M_0$ on the line segment $\arg \epsilon = -\pi/2\alpha$, $0 < |\epsilon| < \rho$, and $|\phi_2(\epsilon)| \leq M_0$ on the line segment $\arg \epsilon = \pi/2\alpha$, $0 < |\epsilon| < \rho$, for some $M_0$. Then we claim that $|\phi_j(\epsilon)| \leq M$ in $S_j$ ($j = 1, 2$), respectively, for some positive number $M$ (cf. Theorem 1).

The difficulty in proving this result is due to the fact that $\phi_1$ and $\phi_2$ are two different functions in $S_1 \cap S_2$, and hence a straightforward application of the Phragmén-Lindelöf theorem does not work.

The proof of this result, may be obtained by defining an auxiliary function (as in the proof of the Phragmén-Lindelöf theorem) together with Theorem 2 (cf. §2). Previously, Y. Sibuya [3] proved a result similar to Theorem 2, in the case when $S \cup S_2 \cup \cdots \cup S_n$ is a disk. However, he did not show that $M$ was actually independent of $c_1$ as $c_1$ tends to zero. In the proof of Theorem 1, we shall let $c_1$ of Theorem 2 tend to zero (as in the proof of the Phragmén-Lindelöf theorem). Since $M$ in Theorem 2 is independent of $c_1$ as $c_1$ tends to zero, such a process will work. Notice also that if $\bigcup_{j=1}^n S_j$ is a disc, there are no boundary rays. In this paper, $\bigcup_{j=1}^n S_j$ being a sector, the analysis near the boundary rays becomes another additional difficulty (cf. Proof of Theorem 2 in §3).

Given functions $\delta_1(\epsilon), \ldots, \delta_n(\epsilon)$ which are holomorphic in sectors $S_1, \ldots, S_n$, respectively, $S_1 \cup \cdots \cup S_n = \{ \epsilon: |\arg \epsilon| < \pi/2\alpha, 0 < |\epsilon| < \rho \}$, satisfying certain reasonable assumptions and having relatively poor estimates on the two rays of the
boundary (i.e. \( \arg e = \pm \pi/2a \)) which are assumed to be close to the imaginary axis, we can substantially improve such estimates along the real axis utilizing Theorem 3 (cf. Theorem 3 in §2). The usefulness of our results is due to the fact that, in general, it is very difficult to get a global result all at once. In some cases where the Phragmén-Lindelöf theorem cannot be applied directly, we still can obtain a global result by putting suitable local results together through an application of our theorem (cf. e.g. [1 and 3]).

2. The statement of theorems.

Theorem 1. Let \( S_j = \{ e: a_j < \arg e < b_j, \ 0 < |e| < \rho \} \) (\( j = 1, \ldots, v \)) be sectors in the right half complex \( e \)-plane where \( \rho > 0 \):
\[
-\pi/2a = a_1 < a_2 < b_1 < a_3 < b_2 < a_4 < b_3 < \cdots < a_v < b_{v-1} < b_v = \pi/2a, \quad a > 1.
\]
Let \( \phi_1(e), \ldots, \phi_v(e) \) be functions of \( e \). Assume that
1. \( \phi_j(e) \) is holomorphic in \( S_j \) and continuous on \( S^*_j = \{ e: a_j \leq \arg e \leq b_j, \ 0 < |e| < \rho \} \),
2. \( |\phi_j(e)| \leq A \exp(c/|e|) \) in \( S_j \), for some positive numbers \( A \) and \( c \),
3. \( |\phi_{j+1}(e) - \phi_j(e)| \leq M_0 \) on the line segment \( \arg e = -\pi/2a, 0 < |e| < \rho \), and \( |\phi_v(e)| \leq M_0 \) on the line segment \( \arg e = \pi/2a, 0 < |e| < \rho \), for some positive number \( M_0 \). Then, there exists a positive number \( M \) such that
\[
|\phi_j(e)| \leq M \text{ in } S_j, \quad j = 1, \ldots, v.
\]

Remark. The inequalities (2.1) mean that \( S_j \cap S_k \neq \emptyset \) only for \( k = j - 1 \) or \( j + 1, j = 2, 3, \ldots, v - 1 \).

Theorem 2. Let \( S_j = \{ e: a_j < \arg e < b_j, \ 0 < |e| < \rho \} \) (\( j = 1, 2, \ldots, v \)) be sectors in the complex \( e \)-plane, where \( \rho > 0 \):
\[
-\pi < a_1 < a_2 < b_1 < a_3 < b_2 < a_4 < b_3 < \cdots < a_v < b_{v-1} < b_v < \pi.
\]
Let \( \delta_1(e), \ldots, \delta_v(e) \) be functions of \( e \). Assume that
1. \( \delta_j(e) \) is holomorphic in \( S_j \), continuous on \( S^*_j \),
2. \( \delta_j(e) \) is asymptotically zero as \( e \) tends to zero in \( S_j \), i.e. \( |\delta_j(e)| \leq K_N |e|^N \) \( (N = 0, 1, 2, \ldots) \) in \( S_j \) for some positive numbers \( K_N \),
3. \( |\delta_{j+1}(e) - \delta_j(e)| \leq M_0 \exp(-c_1/|e|^\lambda) \) in \( S_j \cap S_{j+1} \), \( |\delta_1(e)| \leq M_0 \exp(-c_1/|e|^\lambda) \) on the line segment \( \arg e = a_1, 0 < |e| < \rho \), and \( |\delta_v(e)| \leq M_0 \exp(-c_1/|e|^\lambda) \) on the line segment \( \arg e = b_v, 0 < |e| < \rho \), for some positive numbers \( c_1, M_0 \) and \( \lambda \). Then, there exists a positive number \( M \) which is independent of \( c_1 \) as \( c_1 \) tends to zero such that
\[
|\delta_j(e)| \leq M \exp(-c_1/|e|^\lambda) \quad \text{in } S_j, \quad j = 1, 2, \ldots, v.
\]

Theorem 3. Let \( S_j = \{ e: a_j < \arg e < b_j, \ 0 < |e| < \rho \} \) be sectors in the right half complex \( e \)-plane, where \( \rho > 0 \):
\[
-\pi/2a = a_1 < a_2 < b_1 < a_3 < b_2 < a_4 < b_3 < \cdots < a_v < b_{v-1} < b_v = \pi/2a, \quad a > 1.
\]
Let \( \delta_1(\epsilon), \ldots, \delta_v(\epsilon) \) be functions of \( \epsilon \). Assume that

1. \( \delta_j(\epsilon) \) is holomorphic in \( S_j \), continuous on \( S_j^* \),
2. \( \delta_j(\epsilon) \) is asymptotically zero, as \( \epsilon \) tends to zero in \( S_j \),
3. \(|\delta_j(\epsilon) - \delta_j(\epsilon)| \leq M_0 \exp\{-\mu \Re(1/\epsilon)\} \) in \( S_j \cap S_{j+1} \),

\[ |\delta_j(\epsilon)| \leq M_0 \exp\{-\mu \Re(1/\epsilon)\} \]

on the line segment \( \arg \epsilon = -\pi/2\alpha \), \( 0 < |\epsilon| < \rho \), and

\[ |\delta_j(\epsilon)| \leq M_0 \exp\{-\mu \Re(1/\epsilon)\} \]

on the line segment \( \arg \epsilon = \pi/2\alpha \), \( 0 < |\epsilon| < \rho \), for some positive numbers \( \mu \) and \( M_0 \).

Then, there exists a positive number \( M \) such that \(|\delta_j(\epsilon)| \leq M \exp\{-\mu \Re(1/\epsilon)\} \) in \( S_j \), \( j = 1, 2, \ldots, v \).

3. Proof of the theorems.

(1) Proof of Theorem 2. We denote by \( V_j \) the intersections \( S_j \cap S_{j+1} \), \( j = 1, 2, \ldots, v - 1 \), respectively, and consider an open sector

\[ S = \{ \epsilon: a_i < \arg \epsilon < b_j, 0 < |\epsilon| < \rho_0 \} \]

where \( 0 < \rho_0 < \rho \).

We choose \( v - 1 \) line segments \( l_1, l_2, \ldots, l_{v-1} \) such that \( l_j \subset V_j \), i.e. \( l_j: \epsilon = te^{i\alpha_j} \), \( 0 < t < \rho_0 \), for some \( \alpha_j \) in \( (a_{j+1}, b_j) \). These \( v - 1 \) line segments divide the open sector \( S \) (cf. (3.1)) into \( v \) open sectors \( \hat{S}_1, \hat{S}_2, \ldots, \hat{S}_v \) (cf. Figure 2).
Let \( \gamma_j \ (j = 1, 2, \ldots, \nu) \) be the circular arcs which are defined by \( \epsilon = \rho_0 e^{i \xi} \ (\alpha_{j-1} \leq \xi < \alpha_j) \), respectively, where \( \alpha_0 = a_1, \alpha_\nu = b_\nu \). Set, \( l_0: \epsilon = t e^{i \alpha_1} (0 < t < \rho_0), \ l_\nu: \epsilon = t e^{i \beta_\nu} (0 < t < \rho_0) \). Then \( \gamma_1 + \gamma_2 + \cdots + \gamma_\nu = C = \{ \epsilon: |\epsilon| = \rho_0, a_1 \leq \text{arg} \ \epsilon \leq b_\nu \} \).

The boundaries of \( \tilde{S}_1, \tilde{S}_2, \ldots, \tilde{S}_\nu \) are, respectively, \( l_{j-1} + \gamma_j - l_j, j = 1, 2, \ldots, \nu \).

For \( \epsilon \in \tilde{S}_1 \cup \tilde{S}_2 \cup \cdots \cup \tilde{S}_\nu \), set \( \delta(\epsilon) = \delta_j(\epsilon) \) if \( \epsilon \in \tilde{S}_j \). Since

\[
\frac{1}{2\pi i} \int_{l_{j-1} + \gamma_j - l_j} \frac{\delta_j(\xi)}{\xi - \epsilon} \, d\xi = \begin{cases} \delta_j(\epsilon), & \epsilon \in \tilde{S}_j, \\ 0, & \epsilon \notin \tilde{S}_j, \end{cases}
\]

we have

\[
\delta(\epsilon) = \frac{1}{2\pi i} \sum_{j=1}^\nu \int_{l_{j-1} + \gamma_j - l_j} \frac{\delta_j(\xi)}{\xi - \epsilon} \, d\xi \quad \text{in} \quad \tilde{S}_1 \cup \tilde{S}_2 \cup \cdots \cup \tilde{S}_\nu.
\]

Utilizing \( 1/(\xi - \epsilon) = \sum_{m=0}^N \xi^{-(m+1)} e^m + e^{N+1}/\xi^{N+1}(\xi - \epsilon) \), we derive

\[
\delta(\epsilon) = \frac{1}{2\pi i} \sum_{m=0}^N \left\{ \sum_{j=1}^\nu \int_{l_{j-1} + \gamma_j - l_j} \xi^{-(m+1)} \delta_j(\xi) \, d\xi \right\} e^m + \left\{ \frac{1}{2\pi i} \sum_{j=1}^\nu \int_{l_{j-1} + \gamma_j - l_j} \frac{\delta_j(\xi)}{\xi^{N+1}(\xi - \epsilon)} \, d\xi \right\} e^{N+1}.
\]

Since \( \delta(\epsilon) \) is asymptotically zero as \( \epsilon \) tends to zero in \( \tilde{S}_1 \cup \tilde{S}_2 \cup \cdots \cup \tilde{S}_\nu \), the first term of (3.2) must be zero. Hence

\[
\delta(\epsilon) = \left\{ \frac{1}{2\pi i} \sum_{j=1}^\nu \int_{l_{j-1} + \gamma_j - l_j} \frac{\delta_j(\xi)}{\xi^{N+1}(\xi - \epsilon)} \, d\xi \right\} e^{N+1}.
\]

Therefore, we arrive at the following formula:

\[
\delta(\epsilon) = \frac{1}{2\pi i} \left\{ \int_{l_0} \frac{\delta_1(\xi)}{\xi^{N}(\xi - \epsilon)} \, d\xi - \int_{l_\nu} \frac{\delta_\nu(\xi)}{\xi^{N}(\xi - \epsilon)} \, d\xi \right. \\
+ \sum_{j=1}^{\nu-1} \int_{l_j} \frac{\sigma_j(\xi)}{\xi^{N}(\xi - \epsilon)} \, d\xi + \int_{c} \frac{\delta(\xi)}{\xi^{N}(\xi - \epsilon)} \, d\xi \left\} e^N
\]

for \( \epsilon \in \tilde{S}_1 \cup \tilde{S}_2 \cup \cdots \cup \tilde{S}_\nu \) and \( N = 1, 2, \ldots \) where \( \sigma_j = \delta_{j+1} - \delta_j \). Construct \( \nu \) open sectors \( \bar{S}_1, \bar{S}_2, \ldots, \bar{S}_\nu \) such that the boundaries of \( \bar{S}_j \) are \( \Omega_j \cup T_j \cup \Omega' \), where

\[
\Omega_j = \{ \epsilon: \epsilon = |\epsilon| e^{i(\alpha_{j-1} + \theta)}, |\epsilon| \leq \rho_1 \},
\]

\[
\Omega' = \{ \epsilon: \epsilon = |\epsilon| e^{i(\alpha_j - \theta)}, |\epsilon| \leq \rho_1 \},
\]

\[
T_j = \{ \epsilon: \epsilon = \rho_1 e^{i \theta}, \alpha_{j-1} + \theta \leq \beta \leq \alpha_j - \theta \}, \quad j = 1, 2, \ldots, \nu,
\]

\( 0 < \rho_1 < \rho_0 \) and \( \theta \) is a small positive number such that \( \lambda \theta < \pi \) (cf. Figure 3).
Then, $\tilde{S}_j \subset S_j, j = 1, 2, \ldots, \nu$.

For $\varepsilon \in \tilde{S}_1 \cup \tilde{S}_2 \cup \cdots \cup \tilde{S}_\nu$,

$$\left| \int_{\mathcal{C}_0} \frac{\delta(\xi)}{\xi^N(\xi - \varepsilon)} d\xi \right| \leq \int_{\mathcal{C}} \frac{\delta(\xi)}{|\xi|^N|\xi - \varepsilon|} |d\xi| \leq \int_{\mathcal{C}} \frac{K_\delta}{\rho_0^N(\rho_0 - \rho_1)} |d\xi|$$

$$= \frac{K_\delta \rho_0}{\rho_0^N(\rho_0 - \rho_1)} (b_\nu - a_1) = \frac{B_\delta}{\rho_0^{N-1}}.$$ 

where $K_\delta = \max_{\xi \in \mathcal{C}} |\delta(\xi)|$, $B_\delta = K_\delta(b_\nu - a_1)/(\rho_0 - \rho_1)$.

Similarly,

$$\left| \int_{\mathcal{C}_0} \frac{\delta(\xi)}{\xi^N(\xi - \varepsilon)} d\xi \right| < \frac{M_0}{\sin \theta} \int_0^{\rho_0} t^{-N-1} \exp(-c_1/\tau) d\tau$$

$$\leq \frac{M_0}{\lambda \sin \theta} \int_0^{+\infty} \tau^{(N/\lambda)-1} \exp(-c_1 \tau) d\tau$$

$$= \frac{M_0}{\lambda \sin \theta} C_1^{-(N/\lambda)} \Gamma(N/\lambda).$$
and
\[ \left| \int \frac{\sigma_j(x)}{x^N(x - \varepsilon)} \, dx \right| < \frac{M_0}{\lambda \sin \theta} \frac{C_1^{-(N/\lambda)}}{e^{-(N/\lambda)}} \Gamma(N/\lambda), \quad j = 1, 2, \ldots, \nu - 1. \]

Since \( \Gamma(N/\lambda) \leq M_1(N/\lambda)^{(N/\lambda)} e^{-(N/\lambda)} \) for some \( M_1 > 0 \), we have
\[ \left| \int \frac{\delta_i(x)}{x^N(x - \varepsilon)} \, dx \right| < \frac{M_0 M_1}{\lambda \sin \theta} \left( \frac{N}{c_1 \lambda} \right)^{(N/\lambda)} e^{-(N/\lambda)}, \]
\[ \left| \int \frac{\delta_j(x)}{x^N(x - \varepsilon)} \, dx \right| < \frac{M_0 M_1}{\lambda \sin \theta} \left( \frac{N}{c_1 \lambda} \right)^{(N/\lambda)} e^{-(N/\lambda)} \]
and
\[ \left| \int \frac{\sigma_j(x)}{x^N(x - \varepsilon)} \, dx \right| < \frac{M_0 M_1}{\lambda \sin \theta} \left( \frac{N}{c_1 \lambda} \right)^{(N/\lambda)} e^{-(N/\lambda)} \]
for \( \varepsilon \in \tilde{S}_1 \cup \cdots \cup \tilde{S}_v, j = 1, 2, \ldots, \nu - 1 \). Then,
\[ |\delta(\varepsilon)| \leq \frac{1}{2\pi} \left( \frac{(m + 1)M_0 M_1}{\lambda \sin \theta} \left( \frac{N}{c_1 \lambda} \right)^{(N/\lambda)} e^{-(N/\lambda)} + \frac{B_\delta}{\rho_0^{N-1}} \right) |\varepsilon|^N \]
\[ = \frac{M_2}{2\pi} \left( 1 + \frac{B_\delta \rho_0}{M_2} \left( \frac{c_1 \lambda e}{N \rho_0} \right)^{(N/\lambda)} \right) \left( \frac{N|\varepsilon|^\lambda}{c_1 \lambda} \right)^{(N/\lambda)} e^{-(N/\lambda)}, \]
where \( M_2 = (m + 1)M_0 M_1/\lambda \sin \theta > 0 \).

Since \( \left( 1 + (B_\delta \rho_0/M_2)(c_1 \lambda e/N \rho_0)^{(N/\lambda)} \right) \to 1 \) as \( c_1 \to 0 \), we have
\[ |\delta(\varepsilon)| \leq M_3(\delta) \left( \frac{N|\varepsilon|^\lambda}{c_1 \lambda} \right)^{(N/\lambda)} e^{-(N/\lambda)} \]
for \( \varepsilon \in \tilde{S}_1 \cup \tilde{S}_2 \cup \cdots \cup \tilde{S}_v \), where \( M_3(\delta) \) is a constant which is independent of \( c_1 \) as \( c_1 \) tends to zero.

For a given \( \varepsilon \), choose \( N \) so that \( N/\lambda < c_1/|\varepsilon|^\lambda \leq (N + 1)/\lambda \). We have
\[ \frac{N|\varepsilon|^\lambda}{c_1 \lambda} < 1 \quad \text{and} \quad -\frac{c_1}{|\varepsilon|^\lambda} \geq -\frac{N + 1}{\lambda}. \]

Then, it follows from (3.3) and (3.4) that we have
\[ |\delta(\varepsilon)| \leq M_3(\delta) |\varepsilon|^{\theta/\lambda} \exp \left( -c_1/|\varepsilon|^\lambda \right) \leq M_4 \exp \left( -c_1/|\varepsilon|^\lambda \right), \]
where \( M_4 = \max \{ M_0, M_3(\delta) |\varepsilon|^{\theta/\lambda} \} \) which is independent of \( c_1 \) as \( c_1 \) tends to zero.

Choosing \( l_1, l_2, \ldots, l_{\nu - 1} \) in various ways, we can prove that
\[ |\delta_j(\varepsilon)| \leq M_4 \exp \left( -c_1/|\varepsilon|^\lambda \right) \quad \text{in } S_j', j = 2, \ldots, \nu - 1, \]

where

\[ S'_j = \{ \varepsilon: a_j < \arg \varepsilon < b_j, 0 < |\varepsilon| < \rho_j \} \quad j = 1, 2, \ldots, v. \]

Since \( l_0 \) and \( l_\nu \) are boundaries of \( S \), they cannot be moved. To obtain a similar estimate in \( S'_j \), set \( c_2 = \frac{c_1}{\cos(\lambda \theta/2)} \), then

\[ |\delta_i(\varepsilon e^{i(\alpha_i + \theta/2)})| \leq M_4 \exp(-c_2 \cos(\lambda \theta/2)/|\varepsilon|^\lambda) \]
on \[ \{ \varepsilon: \varepsilon = |\varepsilon| e^{i\theta/2}, 0 < |\varepsilon| < \rho_j \} \cup \{ \varepsilon: \varepsilon = |\varepsilon| e^{-i\theta/2}, 0 < |\varepsilon| < \rho_j \}. \]

Let \( h(\varepsilon) = \exp(c_2/|\varepsilon|^\lambda) \quad (\varepsilon \in \{ \varepsilon: 0 < |\varepsilon| = \rho_j \}). \) For \( \arg \varepsilon = \theta/2, \arg \varepsilon = -\theta/2 \), we have

\[ |\delta_i(\varepsilon e^{i(\alpha_i + \theta/2)})h(\varepsilon)| \leq M_4 \exp\left\{ -c_2 \cos(\lambda \theta/2)/|\varepsilon|^\lambda \right\} \cdot \exp\left\{ -c_2 \cos(\lambda \theta/2)/|\varepsilon|^\lambda \right\} = M_4 \]

and for \( \varepsilon = |\varepsilon| e^{i\theta/2}, -\theta/2 < \eta < \theta/2 \), we have

\[ |\delta_i(\varepsilon e^{i(\alpha_i + \theta/2)})h(\varepsilon)| \leq E \exp\left\{ c_2 \cos(\lambda \eta)/|\varepsilon|^\lambda \right\} \leq E \exp\left\{ c_2/|\varepsilon|^\lambda \right\}, \]

where \( E \) is a bound for \( |\delta_i(\varepsilon e^{i(\alpha_i + \theta/2)})| \) on \( \{ \varepsilon: -\theta/2 < \arg \varepsilon < \theta/2, 0 < |\varepsilon| \leq \rho_j \}. \)

Then, by the Phragmén-Lindelöf theorem, we have

\[ |\delta_i(\varepsilon e^{i(\alpha_i + \theta/2)})h(\varepsilon)| \leq M_4 \exp\left\{ -c_2 \cos(\lambda \theta/2)/|\varepsilon|^\lambda \right\} \]

for \( \varepsilon \in \{ \varepsilon: -\theta/2 < \arg \varepsilon < \theta/2, 0 < |\varepsilon| \leq \rho_j \}\)

i.e. \( |\delta_i(\varepsilon e^{i(\alpha_i + \theta/2)})| \leq M_4 \exp\left\{ -c_2 \cos(\lambda \theta/2)/|\varepsilon|^\lambda \right\} \) for

\[ \varepsilon = |\varepsilon| e^{i\theta/2}, \quad \theta/2 < \arg \varepsilon < \theta/2, \quad 0 < |\varepsilon| \leq \rho_j. \]

Thus, \( |\delta_i(\varepsilon)| \leq M_4 \exp(-c_1/|\varepsilon|^\lambda) \) in \( S'_j \); similarly, \( |\delta_i(\varepsilon)| \leq M_4 \exp(-c_1/|\varepsilon|^\lambda) \) in \( S'_j \).

Choosing \( M \) as the maximal value of \( M_4 \) and a bound of \( |\delta_i(\varepsilon)| \exp(c_1/|\varepsilon|^\lambda) \) on \( \{ \varepsilon: a_j < \arg \varepsilon < b_j, 0 < |\varepsilon| \leq \rho_j \}, j = 1, 2, \ldots, v \), which is independent of \( c_1 \) as \( c_1 \) tends to zero, we can obtain \( |\delta_i(\varepsilon)| \leq M \exp(-c_1/|\varepsilon|^\lambda) \) in \( S'_j \), \( j = 1, 2, \ldots, v \).

(2) PROOF OF THEOREM 1. For each \( \eta > 0 \) we define an auxiliary function

\[ h_\eta(\varepsilon) = \exp\left\{ -\eta/|\varepsilon|^\lambda \right\} \quad (\varepsilon \in \{ \varepsilon: -\pi/2 \alpha \leq \arg \varepsilon \leq \pi/2 \alpha, 0 < |\varepsilon| \leq \rho \}) \]

where \( 1 < \lambda < \alpha \). Set \( \delta_i(\varepsilon) = \phi_j(\varepsilon)h_\eta(\varepsilon) \) which depends on \( \eta, j = 1, 2, \ldots, v \). Then, for \( \varepsilon \in l_0 \),

\[ |\delta_i(\varepsilon)| = |\phi_i(\varepsilon)h_\eta(\varepsilon)| \leq M_0 \exp\left\{ -\eta/|\varepsilon|^\lambda e^{-\pi \lambda/2 \alpha} \right\} = M_0 \exp\left\{ -\eta \cos(\lambda \pi/2 \alpha)/|\varepsilon|^\lambda \right\}; \]

for \( \varepsilon \in l_\nu \),

\[ |\delta_i(\varepsilon)| = |\phi_i(\varepsilon)h_\eta(\varepsilon)| \leq M_0 \exp\left\{ -\eta/|\varepsilon|^\lambda e^{\pi \lambda/2 \alpha} \right\} = M_0 \exp\left\{ -\eta \cos(\lambda \pi/2 \alpha)/|\varepsilon|^\lambda \right\}; \]

and for \( \varepsilon = |\varepsilon| e^{i\beta_j} \in S_j \cap S_{j+1} \),

\[ |\delta_{j+1}(\varepsilon) - \delta_j(\varepsilon)| = |\phi_{j+1}(\varepsilon) - \phi_j(\varepsilon)| |h_\eta(\varepsilon)| \leq M_0 \exp\left\{ -\eta \cos(\lambda |\beta_j|)/|\varepsilon|^\lambda \right\}. \]
Since $0 < \lambda |\beta_j| < \lambda \pi /2\alpha$, we have $\cos(\lambda |\beta_j|) > \cos(\lambda \pi /2\alpha)$. Then we derive

$$|\delta_{j+1}(\epsilon) - \delta_j(\epsilon)| \leq M_0 \exp\left\{-\eta \cos(\lambda \pi /2\alpha) / |\epsilon|^{\lambda}\right\} \quad \text{in } S_j \cap S_{j+1},$$

and $\delta_j(\epsilon)$ is asymptotically zero as $\epsilon$ tends to zero in $S_j$. Set $c_1 = \eta \cos(\lambda \pi /2\alpha)$. Then by Theorem 2, there exists a positive number $H$ which is independent of $c_1$ as $c_1$ tends to zero ($c_1 \to 0$, as $\eta \to 0$) such that

$$|\delta_j(\epsilon)| = |\phi_j(\epsilon)h_{\eta}(\epsilon)| \leq M \exp\left\{-\eta \cos(\lambda \pi /2\alpha) / |\epsilon|^{\lambda}\right\} \quad \text{in } S_j, j = 1, 2, \ldots, \nu.$$

As $\eta \to 0$, $h_{\eta}(\epsilon) \to 1$ for every $\epsilon$; so we conclude that $|\phi_j(\epsilon)| \leq M$ in $S_j, j = 1, 2, \ldots, \nu$.

(3) **Proof of Theorem 3.** Let $h(\epsilon) = \exp\{\mu /\epsilon\}, \epsilon \in S = \{\epsilon: -\pi /2\alpha \leq \arg \epsilon \leq \pi /2\alpha, 0 < |\epsilon| \leq \rho\}$, and set $\phi_j(\epsilon) = \delta_j(\epsilon)h(\epsilon), j = 1, 2, \ldots, \nu$. Then, for $\epsilon = |\epsilon| e^{i\pi /2\alpha}$,

$$|\phi_j(\epsilon)| = |\delta_j(\epsilon)| |h(\epsilon)| \leq M_0 \exp\left\{-\mu \text{Re}(1/\epsilon)\right\} \exp\{\mu \text{Re}(1/\epsilon)\} = M_0,$$

similarly, for $\epsilon = |\epsilon| e^{i\pi /2\alpha}, |\phi_j(\epsilon)| \leq M_0$; for $\epsilon = |\epsilon| e^{i\beta_j} \in S_j \cap S_{j+1}, |\phi_{j+1}(\epsilon) - \phi_j(\epsilon)| \leq M_0$; and for $\epsilon = |\epsilon| e^{i\xi_j} \in S_j$,

$$|\phi_j(\epsilon)| = |\delta_j(\epsilon)| |h(\epsilon)| \leq A \exp\left\{\mu \cos \xi_j / |\epsilon|\right\} \leq A \exp\{\mu /|\epsilon|\}$$

where $A > 0$ is a bound for $\delta_j$ in $S_j$. By Theorem 1, we have $|\phi_j(\epsilon)| \leq M$ in $S_j$ ($j = 1, 2, \ldots, \nu$), i.e. $|\delta_j(\epsilon)| \leq M \exp\{-\mu \text{Re}(1/\epsilon)\}$ in $S_j, j = 1, 2, \ldots, \nu$.

**Acknowledgement.** I would like to thank my advisor Professor Sibuya for his many valuable suggestions.

**References**

