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UNIQUENESS OF TAYLOR'S FUNCTIONAL CALCULUS

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ABSTRACT. Two uniqueness results concerning Fréchet module structures over algebras of holomorphic functions defined on some complex manifolds are presented, containing as particular cases uniqueness theorems for J. L. Taylor's analytic functional calculi for commuting n-tuples of linear continuous operators on Fréchet spaces [7], [9]. Namely, the first statement says that the Spectral Mapping Theorem insures the unicity of the functional calculus and thus it improves Zame's unicity theorem [11, Theorem 1], while the second statement gives a unicity condition which is an analogue of the compatibility property [3, Theorem I.4.1] in spectral theory of several variables in commutative Banach algebras. As a corollary the two functional calculi constructed in [7] and [9] by J. L. Taylor coincide.

0. Introduction. This note is a continuation of [5] and [6] and is presented in the spirit of and with the technique initiated by J. L. Taylor in [9]. The general philosophy is to regard a commuting n-tuple a of linear continuous operators on a Fréchet space M as a Fréchet module structure of M over the algebra $\mathcal{O}(\mathbf{C}^n)$ of entire functions and then to use homological-topological methods.

Proposition 1 essentially contains in its statement Fréchet module structures over algebras of holomorphic functions on complex manifolds which are not canonically imbedded in a numerical space, and thus it is quite difficult to give it an interpretation in terms of n-tuples of commuting operators. On the other hand, Theorem 2 has a clear specialization in the operator-theoretic language as follows:

THEOREM 1. *For each open subset U of $\mathbf{C}^n$ which contains the joint spectrum of an n-tuple a of linear continuous commuting operators on a Banach space, there is exactly one continuous functional calculus for a with analytic functions on U which satisfies the spectral mapping theorem.*

In fact this statement holds for Fréchet spaces with the additional assumption that the n-tuple has a continuous functional calculus with entire functions. The correspondence with Theorem 1 of [11], which asserts that an n-tuple of elements of a Banach algebra has exactly one continuous functional calculus with germs of analytic functions in neighbourhoods of the joint spectrum and which commutes with the Gel'fand transformation, is now obvious.

The statements below are presented in the spirit of [5] and [6], without distinguished coordinates, and their form is imposed by the restriction that the open

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sets in question must have an envelope of holomorphy. The homological-topological technique initiated by J. L. Taylor in problems of functional calculus in [**8**] and [**9**] is again used, but even Theorem 1 needs for its proof the special results of transversality from [**2**].

Let us recall some facts and notation from [**5**], [**6**]. Each Fréchet $\mathcal{O}(X)$-module M over the nuclear Fréchet algebra $\mathcal{O}(X)$ of global sections of a Stein space X has a distinguished closed subset $\sigma(X, M)$ of X, with a good homological characterization in terms of topological Tor's, which is the generalization of Taylor's joint spectrum of an n-tuple of commuting operators. If $f\colon X \to Y$ is a morphism of complex spaces and M is a Fréchet $\mathcal{O}(X)$-module, then the $\mathcal{O}(Y)$-module M with the structure induced by $f^*\colon \mathcal{O}(Y) \to \mathcal{O}(X)$ will be denoted by M^f.

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1. The main result. It is known [**9**], [**5**], [**6**] that a Fréchet $\mathcal{O}(X)$-module M, with X a finite-dimensional Stein space, has a *natural extension* to a Fréchet module structure over the topological algebra $\mathcal{O}(\sigma(X, M))$ of germs of analytic functions defined in neighbourhoods of $\sigma(X, M)$. Moreover, the set $\sigma(X, M)$ and the natural extension are functorial in some precise sense [**5**, Theorem 5], [**6**, Theorems 1 and 2].

We shall be interested in a converse question; namely, to give minimal functorial properties on an abstract extension of a Fréchet module structure from $\mathcal{O}(X)$ to $\mathcal{O}(\sigma(X, M))$ which insures its coincidence with the natural extension.

Let us remark that if the Fréchet $\mathcal{O}(X)$-module M has a Fréchet $\mathcal{O}(\sigma(X, M))$-module structure, then *a fortiori* M is a Fréchet $\mathcal{O}(U)$-module for every open subset U of X containing $\sigma(X, M)$.

THEOREM 2. *Let X be a Riemann domain which is a Stein manifold, let M be a Fréchet space and consider an arbitrary extension of a given Fréchet $\mathcal{O}(X)$-module structure on M to a Fréchet $\mathcal{O}(U)$-module structure, denoted M^U, for an open subset $\sigma(X, M) \subset U \subset X$.*

If for every holomorphic map $f\colon U \to \mathbf{C}^n$, $n \geq 1$, the equality

$$\text{(i)} \qquad \overline{f\sigma(X, M)} = \sigma\left(\mathbf{C}^n, (M^U)^f\right)$$

holds, then M^U coincides with the Fréchet $\mathcal{O}(U)$-module structure of M given by the natural extension.

PROOF. Let us denote by M_1^U the natural extension to a Fréchet $\mathcal{O}(U)$-module structure given by the isomorphism [**9**], [**5**]:

$$(1) \qquad M_1^U = \hat{\mathrm{Tor}}_0^{\mathcal{O}(X)}(\mathcal{C}^{\cdot}(\mathfrak{U}), M),$$

where $\mathfrak{U}$ is an open Stein covering of $U = \bigcup_{i \in I} U_i$ with finite-dimensional nerve, and $\mathcal{C}^{\cdot}(\mathfrak{U})$ is the associated Čech complex.

Let $\tilde{U}$ be the envelope of holomorphy of U which exists by [**4**, Theorem 5.4.5] and is a Stein manifold with $\dim \tilde{U} = \dim X$. Regarding M^U as a Fréchet $\mathcal{O}(\tilde{U})$-module we have an isomorphism of Fréchet spaces,

$$M^U = \hat{\mathrm{Tor}}_0^{\mathcal{O}(\tilde{U})}(\mathcal{C}^\cdot(\mathfrak{U}), M^U), \tag{2}$$

because $\sigma(\tilde{U}, M^U) \subset U$ and $\mathfrak{U}$ remains an open Stein covering of U in $\tilde{U}$.

Indeed, if $f\colon \tilde{U} \to \mathbf{C}^n$, $n = 2\dim(X) + 1$, is a closed imbedding (see [**4**, Theorem 5.3.9]), then by [**6**, Theorem 1]

$$\overline{f\sigma(\tilde{U}, M^U)} = \sigma\big(\mathbf{C}^n, (M^U)^f\big) \tag{3}$$

and condition (i) compared with (3) yields $\sigma(\tilde{U}, M^U) = \sigma(X, M) \subset U$.

The right side of (3) is computed as the zero homology spaces of the simple complex associated to the bicomplex $\tilde{B}^{\mathcal{O}(\tilde{U})}_{\cdot}(\mathcal{C}^\cdot(\mathfrak{U}), M^U)$, see [**5**], and each element m of M^U is represented as the class of $1 \otimes m \in \mathcal{C}^0(\mathfrak{U}) \hat{\otimes} M^U$ in that complex. The operation that gives $\hat{\mathrm{Tor}}_0$ allows us to identify $\varphi \otimes m$ with $1 \otimes \varphi m$ as elements of $\mathcal{C}^0(\mathfrak{U}) \hat{\otimes} M^U$, where $\varphi \in \mathcal{O}(\tilde{U})$, so that (2) is even an isomorphism of (left) $\mathcal{O}(\tilde{U})$-modules.

But this means that the natural morphism induced by the extension of the inclusion $U \subset X$,

$$M_1^U \cong \hat{\mathrm{Tor}}_0^{\mathcal{O}(X)}(\mathcal{C}^\cdot(\mathfrak{U}), M) \to \hat{\mathrm{Tor}}_0^{\mathcal{O}(\tilde{U})}\big(\mathcal{C}^\cdot(\mathfrak{U}), M^U\big) \cong M^U,$$

is an isomorphism of $\mathcal{O}(U)$-modules. Q.E.D.

Note that condition (i) can be required only for holomorphic maps $f\colon U \to \mathbf{C}^{2\dim(X)+1}$.

2. The functorial selection. We shall now present another uniqueness result which highlights the difference between the spectral theory in commutative Banach algebras and that on Banach (or Fréchet) spaces, namely, Arens and Calderón's lemma from Banach algebra [**1**], [**3**] seems to have a natural analogue in the spectral theory on Banach spaces which is obtained by passing from an open set to its envelope of holomorphy.

DEFINITION. Let M be a Fréchet space and $\mathcal{C}$ a category of finite-dimensional Stein spaces and morphisms of analytic spaces.

A *functorial selection* of extensions of Fréchet module structures on M is an assignment from each Fréchet $\mathcal{O}(X)$-module structure on M, $X \in \mathcal{O}b\,\mathcal{C}(\mathcal{C})$, to an extension to a Fréchet $\mathcal{O}(U)$-module structure on M, denoted M^U, where U is an open subset of X which contains $\sigma(X, M)$, so that

For every morphism $f\colon X \to Y$ in $\mathcal{C}$, every structure of a Fréchet $\mathcal{O}(X)$-module on M and all open sets $\sigma(X, M) \subset U \subset X$, $\sigma(Y, M^f) \subset V \subset Y$, $f(U) \subset V$, one has

$$(M^U)^f = (M^f)^V. \tag{ii}$$

Note that a functorial selection endows each Fréchet $\mathcal{O}(X)$-module M with a Fréchet module structure over $\mathcal{O}(\sigma(X, M))$ because condition (ii) applied for the identical map gives a direct system of structures over the open neighbourhoods of $\sigma(X, M)$.

PROPOSITION 1. *There is exactly one functorial selection of extensions of Fréchet module structures on a Fréchet space M in the category of Riemann domains and Stein manifolds.*

PROOF. Let X be a Riemann domain which is a Stein manifold and consider a Fréchet $\mathcal{O}(X)$-module structure on M. If U is an open subset of X containing $\sigma(X, M)$, then we have the assumed structure of the Fréchet $\mathcal{O}(U)$-module M^U; on the other hand, we have the natural one **[5]**, denoted M_1^U, which is also an extension of the $\mathcal{O}(X)$-structure. But U is a Riemann domain, and hence it has an envelope of holomorphy $\tilde{U}$ [**4**, Theorem 5.4.5]; thus the inclusion map has an extension $f\colon \tilde{U} \to X$. Because $\mathcal{O}(\tilde{U}) = \mathcal{O}(U)$, M_1^U is also a Fréchet $\mathcal{O}(\tilde{U})$-module and, moreover, $(M_1^U)^f = M$.

By using (ii) in that case,

$$\left[\left(M_1^U\right)^{f^{-1}(U)}\right]^f = M^U. \tag{4}$$

Noticing that all holomorphic functions on U and $f^{-1}(U)$ have unique extensions to $\tilde{U}$, (4) means exactly that $M_1^U = M^U$. Q.E.D.

In the commutative Banach algebras framework, Arens and Calderón's lemma allows us to considerably restrict the category of Stein spaces so Theorem I.4.1 of **[3]** can be stated as follows:

PROPOSITION 2. *There is exactly one functorial selection of extensions of Banach-module structures associated to n-tuples of elements of a commutative Banach algebra in the category consisting of numerical spaces* $\mathbf{C}^n$ *and projection maps between them.*

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