A GAP TAUBERIAN THEOREM
FOR GENERALISED ABSOLUTE ABEL SUMMABILITY

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Abstract. A gap Tauberian theorem for generalised absolute Abel summability \(|A_n|\)

is proved using Mel’nik’s theorem on convolution transforms.

1. Introduction. The well-known gap Tauberian theorem for Abel summability

\((A_0) = (A)\) is a special case of the high indices theorem of Hardy and Littlewood

[2, Theorem 114]. The gap Tauberian theorem for \((A_\alpha)\) summability has been proved

by Krishnan [3]. Zygmund [5] has proved that \(|A, \lambda_n|\) summability implies absolute

convergence when \((\lambda_n)\) satisfies the high indices condition \(\lambda_{n+1}/\lambda_n \geq c > 1\), and

Mel’nik [4] had deduced the same result as a corollary of his general theorem, which

is stated here as Lemma 1. The gap Tauberian theorem for absolute Abel summability

\(|A_0| \equiv |A|\) is a special case of Zygmund’s result when we take \((\lambda_n)\) as a sequence

of integers. The purpose of this note is to show that the gap Tauberian theorem for

absolute \(A_\alpha\) summability \(|A_\alpha|\) can be deduced from Mel’nik’s theorem.

2. Definitions and notations. Let \(\alpha > -1\). For a given series \(\sum_{n=0}^{\infty} a_n\), write

\[ A_n = \sum_{r=0}^{n} a_r \quad (n \geq 0), \]

\[ \tilde{A}(y) = \sum_{n \leq y} a_n, \]

\[ a(x) = \sum_{n=0}^{\infty} a_n \left( \frac{n+\alpha}{\alpha} \right) x^n \quad (0 < x < 1), \]

\[ A(x) = \sum_{n=0}^{\infty} A_n \left( \frac{n+\alpha}{\alpha} \right) x^n \quad (0 < x < 1), \]

\[ f(x) = (1-x)^{\alpha+1} A(x) \quad (0 < x < 1). \]

We assume that the series defining \(a(x)\) and \(A(x)\) converge for \(0 < x < 1\). \(\sum a_n\) is

summable \((A_\alpha)\) to \(A\) if

\[ f(x) = (1-x)^{\alpha+1} A(x) \to A \quad \text{as} \ x \to 1-, \]
and absolutely $A_n$ summable to $A$ ($|A_n|$ summable to $A$) if
\[ \int_0^1 \left| \frac{d}{dx} f(x) \right| dx < \infty \]
and
\[ f(x) = (1 - x)^{n+1} A(x) \rightarrow A \quad \text{as} \quad x \rightarrow 1^- . \]
Abel summability and absolute Abel summability are, respectively, $A_0$ and $|A_0|$ summability methods.

We say, following Mel’nik, that the function $s(v)$ belongs to the class $|T_0|$ if (i) $s(v)$ is of bounded variation in every finite interval; (ii) there exist constants $\mu$, $\delta$ and a function $\theta(v)$, depending only on $v$, such that $0 < \mu$, $\delta \leq 1$, and the inequality
\[ \Re \{e^{i\theta} ds(u) \} \geq \mu |ds(u)| \]
is satisfied for all $u$ in $[v - \delta, v + \delta]$.

3. Theorem and lemmas. The main result is

**Theorem.** If $\Sigma a_n$ is summable $|A_n|$ and satisfies a gap condition
(G) $a_n = 0$ for $n \neq n_k$, where $(n_k)$ is a sequence of positive integers such that
$n_0 > 0$, $n_{k+1}/n_k \geq c > 1$ for all $k = 0, 1, 2, \ldots$, then $\Sigma a_n$ converges absolutely.

We first prove some preliminary lemmas.

**Lemma 1.** Let the kernel $k(u)$ of the integral transform
\[ g(v) = \int_{-\infty}^{\infty} k(v - u) ds(u) \]
be Borel measurable,
\[ \sum_{n=-\infty}^{\infty} \sup_{n \leq u < n+1} |k(u)| < \infty , \]
and its Fourier transform
\[ K(t) = \int_{-\infty}^{\infty} e^{-iut} k(u) du \neq 0 \]
for $t \in E = (-\infty, \infty)$. Also assume $s(u)$ belongs to the class $|T_0|$ and
\[ \int_{v}^{v+1} |ds(u)| \leq M \text{ for } v \in E . \]
Then we can find a constant $C$ depending only on $\delta$, $\mu$ and $k(u)$ such that
\[ \int_{-\infty}^{\infty} |s(v)| \leq C \int_{-\infty}^{\infty} |g(v)| dv . \]
This is contained in Mel’nik’s theorem. [4, Theorem 1].

**Lemma 2.** If $\Sigma a_n$ is summable $|A_n|$ and satisfies the gap condition (G), then
\[ \int_0^1 |f(x)| dx = \int_{-\infty}^{\infty} |g_1(v)| dv , \]
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where
\( g_1(v) = \int_{-\infty}^{\infty} k_1(v, u) \, ds(u), \)
\( s(u) = \overline{A}(e^u), \)
\( k_1(v, u) = \begin{cases} \frac{(1 - \exp(e^{-v}))^\alpha e^{-v} \exp(-e^{-v}) \Gamma(\alpha + e^u + 1)}{\Gamma(1 + \alpha) \Gamma(e^u)} & \text{for } u \geq 0, \\ k(v - u) & \text{for } u < 0. \end{cases} \)

and
\( k(t) = \exp\{-(\alpha + 1)t - e^{-t}\}/\Gamma(\alpha + 1). \)

**Proof.** First we find an expression for \( f'(x) \) in terms of \( a_n. \)

\[
f'(x) = -(\alpha + 1)(1 - x)^\alpha \sum_{n=0}^{\infty} \frac{(n + \alpha)}{\alpha} A_n x^n + (1 - x)^{\alpha+1} \sum_{n=0}^{\infty} \frac{(n + \alpha)}{\alpha} A_n x^{n-1}
\]

\[
= (1 - x)^\alpha(\alpha + 1) - \sum_{n=0}^{\infty} \frac{(n + \alpha)}{\alpha} A_n x^n + (1 - x) \sum_{n=1}^{\infty} \frac{(n + \alpha)}{n - 1} A_n x^{n-1}
\]

\[
= (1 - x)^\alpha(\alpha + 1) \left[ -A_n \left( \frac{n + \alpha}{\alpha} + \frac{n + \alpha}{n - 1} \right) + A_{n+1} \left( n + 1 + \alpha \right) \right] x^n
\]

\[
= (1 - x)^\alpha(\alpha + 1) \sum_{n=0}^{\infty} \left( \frac{n + 1 + \alpha}{n} \right) a_{n+1} x^n
\]

\[
= (1 - x)^\alpha(\alpha + 1) \sum_{n=1}^{\infty} \frac{(n + \alpha)}{n - 1} a_n x^{n-1}
\]

\[
= (1 - x)^\alpha(\alpha + 1) \int_1^{\infty} \frac{\Gamma(y + \alpha + 1)}{\Gamma(y) \Gamma(\alpha + 2)} x^{y-1} d\overline{A}(y)
\]

\[
= (1 - x)^\alpha \int_1^{\infty} \frac{\Gamma(y + \alpha + 1)}{\Gamma(y) \Gamma(\alpha + 1)} x^{y-1} d\overline{A}(y).
\]

Substituting \( x = \exp\{-e^{-v}\}, y = e^u \), we obtain

\[
f'(x) = F(v) = \int_0^{\infty} \frac{(1 - \exp(-e^{-v}))^\alpha \Gamma(\alpha + e^u + 1)}{\Gamma(1 + \alpha) \Gamma(e^u)} \frac{\exp(-e^{-v})}{\exp(-e^{-v})} d\overline{A}(e^u).
\]

Hence,
\[
\int_0^1 |f'(x)| \, dx = \int_{-\infty}^{\infty} |F(v)| \frac{dx}{dv} dv = \int_{-\infty}^{\infty} |g_1(v)| \, dv.
\]
where

\[ g_1(v) = F(v) \frac{dx}{dv} \]

\[ = \int_0^\infty \frac{(1 - \exp(-e^{-v}))^\alpha \Gamma(\alpha + e^v + 1)}{\Gamma(1 + \alpha) \Gamma(e^v)} \exp(-e^{-v}) \exp(-e^{-v}) e^{-v} d\bar{A}(e^v) \]

\[ = \int_0^\infty \frac{(1 - \exp(-e^{-v}))^\alpha e^{-v}}{\Gamma(1 + \alpha) \Gamma(e^v)} \exp(-e^{-v}) \Gamma(\alpha + e^v + 1) d\bar{A}(e^v) \]

\[ + \int_0^0 k(v - u) d\bar{A}(e^u), \]

because \( a_0 = 0 \) under the assumption of the gap condition (G), and hence \( \bar{A}(e^u) = 0 \) for \( u < 0 \). Here \( k(t) \) is given by (10). Hence,

\[ \int_0^1 |f'(x)| dx = \int_{-\infty}^{\infty} k_1(v, u) ds(u), \]

where \( s(u) \) and \( k_1(v, u) \) are given by (8) and (9). □

Mel’nik [4, p. 834] has observed that his theorem can be applied to functions \( g_1(v) \) which are expressible as

\[ g_1(v) = \int_{-\infty}^{\infty} k_1(v, u) ds(u), \]

where \( k_1(v, u) \) is not of the canonical form \( k(v - u) \) as in (1) but can be “approximated” to a canonical form in a certain sense. The context of Lemma 3 below is that the kernel \( k_1(v, u) \) appearing in (7) can be approximated in this sense. In the proof of the Theorem in §4 we incorporate the details as to how this approximation is useful.

**Lemma 3.** If \( k(t) \) and \( k_1(v, u) \) are the kernels of Lemma 2 given by (10) and (9), respectively, and \( -1 < \alpha < 0 \), then

\[ \sum_{n=-\infty}^{\infty} \max_{n \leq u < n+1} \int_{-\infty}^{\infty} |k_1(v, u) - k(v - u)| dv = L < \infty. \]

**Proof.** For \( u < 0 \), \( k_1(v, u) = k(v - u) \) and therefore it suffices to prove

\[ \sum_{n=0}^{\infty} \max_{n \leq u < n+1} \int_{-\infty}^{\infty} |k_1(v, u) - k(v - u)| dv = L < \infty. \]

Let \( u \geq 0 \). Then

\[ k_1(v, u) - k(v - u) = \frac{(1 - \exp(-e^{-v}))^\alpha e^{-v} \exp(-e^{v-u}) \Gamma(\alpha + e^v + 1)}{\Gamma(1 + \alpha) \Gamma(e^v)} \]

\[ = \frac{(1 - x)^\alpha (\log x^{-1}) x^y \Gamma(\alpha + y + 1)}{\Gamma(1 + \alpha) \Gamma(y)} - \frac{y^{\alpha+1}(\log x^{-1})^{\alpha+1} x^y}{\Gamma(1 + \alpha)} \]

\[ = x^y \left( \log \frac{1}{x} \right) \left[ \frac{(1 - x)^\alpha \Gamma(\alpha + y + 1)}{\Gamma(y) \Gamma(1 + \alpha)} \right] - \left( \log x^{-1} \right)^{\alpha+1}. \]
Now
\[ \Gamma(\alpha + y + 1)/\Gamma(y) = y^{\alpha + 1} + O(y^\alpha) \text{ uniformly in } y \geq 1. \]

Also
\[ \log x^{-1} = (1 - x) + O(1 - x)^2 \text{ uniformly in } \delta < x < 1 \]

and, since \( \alpha < 0 \), it follows that
\[ (\log x^{-1})^\alpha = (1 - x)^\alpha + O(1 - x)^{\alpha + 1} \]

uniformly in \( 0 < x < 1 \). It follows that, uniformly in \( y \geq 1, 0 < x < 1 \), (12) is
\[ x^y \log x^{-1} \{ O(y^\alpha (1 - x)^\alpha) + O(y^{\alpha + 1} (1 - x)^{\alpha + 1}) \}. \]

Hence, uniformly in \( y \geq 1 \), we have
\[ \int_{-\infty}^{\infty} |k_1(v, u) - k(v - u)| dv = O\left\{ y^\alpha \int_0^1 x^{1 - y} (1 - x)^\alpha dx \right\} + O\left\{ y^{\alpha + 1} \int_0^1 x^{1 - y} (1 - x)^{\alpha + 1} dx \right\} = O(1/y) = O(e^{-u}). \]

The lemma follows.

4. Proof of the Theorem. By Theorems 2 and 5 of [1], summability \(|A_\lambda|\) implies \(|A_\mu|\) for \( \lambda > \mu > -1 \). Hence it suffices to prove the theorem for \( -1 < \alpha < 0 \). Define
\[ g(v) = \int_{-\infty}^{\infty} k(v - u) ds(u) \]

and apply Lemma 1 to \( g(v) \). It has the canonical form (1). (2) can be easily verified. The Fourier transform of \( k(x) \) is
\[ K(t) = \Gamma(\alpha + 1 + it)/\Gamma(\alpha + 1) \neq 0 \]

for any \( t \), hence (3) is satisfied. If (G) is satisfied, and if \( \delta > 0 \) is sufficiently small, it follows that for any \( v \) the interval \([v - \delta, v + \delta]\) contains not more than one point at which the function \( s(u) = \tilde{A}(e^u) \) has a jump. Hence we see that \( s(u) \) belongs to the class \( |T_0| \) where \( \mu = 1, \delta \) depends only on \( c \). Since \(|A_\alpha|\) summability implies \((A_\alpha)\) summability and the gap Tauberian theorem is true for \((A_\alpha)\) summability (vide [3]), \( \sum a_n \) is convergent and hence follows the boundedness of the terms of \( \sum a_n \). In view of (G), (4) now follows. (2) and (4) show that the integral (13) is absolutely convergent. All the requirements of Lemma 1 are satisfied and hence by the conclusion of the same lemma, we obtain
\[ \sum_{n=0}^{\infty} |a_n| \leq C \int_{-\infty}^{\infty} |g(v)| dv. \]

Now
\[ \int_{-\infty}^{\infty} |g(v)| dv \leq \int_{-\infty}^{\infty} |g(v) - g_1(v)| dv + \int_{-\infty}^{\infty} |g_1(v)| dv, \]

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where \( g_1(v) \) is given by (7). Then, substituting for \( g(v) \) and \( g_1(v) \) and applying Lemma 2, we get

\[
\int_{-\infty}^{\infty} |g(v)| \, dv \leq \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |k_1(v, u) - k(v - u)| \, ds(u) \, |dx| + \int_{0}^{1} f'(x) \, dx
\]

\[
\leq \int_{-\infty}^{\infty} |ds(u)| \int_{-\infty}^{\infty} |k_1(v, u) - k(v - u)| \, dv + \int_{0}^{1} f'(x) \, dx
\]

\[
\leq \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} |ds(u)| \max_{n \leq u < n+1} \int_{-\infty}^{\infty} |k_1(v, u) - k(v - u)| \, dv + \int_{0}^{1} f'(x) \, dx
\]

\[
< ML + C' + \int_{0}^{1} f'(x) \, dx \quad \text{(since (4) is satisfied)}
\]

\[
(15) = C' + \int_{0}^{1} f'(x) \, dx.
\]

From (14) and (15), we obtain

\[
\sum_{n=0}^{\infty} |a_n| < C \left[ C' + \int_{0}^{1} f'(x) \, dx \right] = C_1 + C \int_{0}^{1} f'(x) \, dx < \infty.
\]

The theorem is proved.

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References


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