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QUATERNION KAEHLERIAN MANIFOLDS ISOMETRICALLY IMMERSED IN EUCLIDEAN SPACE

M. BARROS AND F. URBANO

ABSTRACT. Let M be a complete $4n$-dimensional quaternion Kaehlerian manifold isometrically immersed in the $(4n+d)$-dimensional Euclidean space. In this note we prove that if $d < n$, then M is a Riemannian product $Q^m \times P$, where Q^m is the m-dimensional quaternion Euclidean space $(m \geqslant n-d)$ and P is a Ricci flat quaternion Kaehlerian manifold.

1. Introduction. In [**2**] C. Fwu obtains some topological restrictions for Kaehler manifolds which can be isometrically immersed in Euclidean space with low codimension. He essentially adapts a technique used by J. D. Moore [**5**] and uses elementary Morse theory. In a second result, he assumes the Kaehler manifold has nonnegative sectional curvature and thus obtains more information about its geometric structure (see [**2**, Theorem 2]).

In this paper we consider quaternion Kaehler manifolds isometrically immersed in Euclidean space with low codimension. We adapt the technique used in [**5**]. Since any quaternion Kaehler manifold with quaternion dimension $n > 1$ is an Einstein space (see [**3**, Theorem 3.3]), we obtain our main result without using any hpothesis about sectional curvature (compare [**2**, Theorems 1 and 2]).

THE MAIN THEOREM. *Let M be a complete quaternion Kaehler manifold of real dimension $4n$. If $\phi\colon M \to E^{4n+d}$ $(d < n)$ is an isometric immersion, then $M = Q^m \times P$, the Riemannian product of Q^m $(m \geqslant n-d)$ and a Ricci flat quaternion Kaehler manifold P of real dimension $4(n-m)$. Moreover $\phi = 1 \times \psi$, the product of the identity map of Q^m onto E^{4m} and an isometric immersion of P into $E^{4(n-m)+d}$ (where E^r (resp. Q^r) denotes the r-dimensional Euclidean space (resp. the r-dimensional quaternion Euclidean space)).*

2. The fundamental lemma. Let M be a quaternion Kaehler manifold, i.e. there exists a 3-dimensional vector bundle V of tensors of type $(1,1)$ with local basis of almost Hermitian structures I, J, K such that: (i) $I \cdot J = -J \cdot I = K$, and (ii) for any local cross-section Ω of V and any local vector field X tangent to M, $\nabla_X \Omega$ is also a local cross-section of V, where ∇ denotes the covariant differentiation on M (see [**3**] for details).

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Let $\langle\ ,\ \rangle$ be the Riemannian metric on M and R its curvature tensor. Then we have, [3]

$$\begin{aligned}(2.1)\qquad &\langle R(X,Y)IZ, IW\rangle - \langle R(X,Y)Z,W\rangle \\ &\qquad = -C(X,Y)\langle KZ,W\rangle - B(X,Y)\langle JZ,W\rangle, \\ &\langle R(X,Y)JZ, JW\rangle - \langle R(X,Y)Z,W\rangle \\ &\qquad = -C(X,Y)\langle KZ,W\rangle - A(X,Y)\langle IZ,W\rangle, \\ &\langle R(X,Y)KZ, KW\rangle - \langle R(X,Y)Z,W\rangle \\ &\qquad = -B(X,Y)\langle JZ,W\rangle - A(X,Y)\langle IZ,W\rangle\end{aligned}$$

for any X, Y, Z, W tangent to M, where A, B, C denote certain local 2-forms over M associated to the local basis $\{I, J, K\}$.

Henceforth, we suppose M is a $4n$-dimensional quaternion Kaehler manifold isometrically immersed in E^{4n+d} $(d < n)$. We denote by σ the second fundamental form of that immersion, and T_pM (resp. $T_p^\perp M$) will denote the tangent space (resp. the normal space) at a point $p \in M$. Let $W_p = T_p^\perp M \oplus T_p^\perp M \oplus T_p^\perp M \oplus T_p^\perp M$ be the direct sum of four copies of $T_p^\perp M$. We consider the indefinite inner product on W_p,

$$\begin{aligned}(2.2)\quad &\langle\langle \xi_1 \oplus \xi_2 \oplus \xi_3 \oplus \xi_4, \eta_1 \oplus \eta_2 \oplus \eta_3 \oplus \eta_4\rangle\rangle \\ &\qquad = 3\langle \xi_1, \eta_1\rangle - (\langle \xi_2, \eta_2\rangle + \langle \xi_3, \eta_3\rangle + \langle \xi_4, \eta_4\rangle),\end{aligned}$$

and define a bilinear map $\beta\colon T_pM \times T_pM \to W_p$ by

$$(2.3)\qquad \beta(x,y) = \sigma(x,y) \oplus \sigma(x,Iy) \oplus \sigma(x,Jy) \oplus \sigma(x,Ky).$$

By using (2.1) and the Gauss equation, we have

$$(2.4)\qquad \langle\langle \beta(x,z), \beta(y,w)\rangle\rangle - \langle\langle \beta(x,w), \beta(y,z)\rangle\rangle = T(x,y,z,w),$$

where

$$T(x,y,z,w) = -2\{A(x,y)\langle Iz,w\rangle + B(x,y)\langle Jz,w\rangle + C(x,y)\langle Kz,w\rangle\}.$$

A priori, β is not flat with respect to $\langle\langle\ ,\ \rangle\rangle$ in the sense of [5].

For each fixed vector $x \in T_pM$, $\beta(x)(y) = \beta(x,y)$ defines a linear map $\beta(x)\colon T_pM \to W_p$. A vector x_0 is said to be left regular if

$$\dim \beta(x_0)(T_pM) = \max\{\dim \beta(x)(T_pM)/x \in T_pM\} = q.$$

Let

$$N(\beta, x) = \{n \in T_pM/\beta(x,n) = 0\}$$

be the kernel of $\beta(x)$.

LEMMA. *Let x_0 be a left regular vector and $N_p = N(\beta, x_0)$. Then:*

(i) *N_p is a quaternionic subspace of T_pM, i.e., $I(N_p) = J(N_p) = K(N_p) = N_p$;*

(ii) $\dim N_p \geq 4(n-d)$;

(iii) $\sigma(n, n') = 0$ *for all $n, n' \in N_p$.*

PROOF. Since the proofs of (i) and (ii) are very easy, we will prove (iii).

By using an argument similar to [2, Lemma, (i)], we see that $\beta(x, n) \in \beta(x_0)(T_pM)$ for all $(x, n) \in T_pM \times N_p$.

Let W_1 be the subspace generated by $\beta(T_p, M, N_p)$ in W_p. We choose any two vectors $(y_i, n_i) \in T_pM \times N_p$, $i = 1, 2$. Then there exists $v_1 \in T_pM$ such that $\beta(x_0, v_1) = \beta(y_1, n_1)$. Let v_1' (resp. v_1'') be the N_p-component (resp. the $N_p^\perp$-component, where $N_p^\perp$ denotes the orthogonal complementary subspace of N_p in T_pM) of v_1. Therefore $\beta(x_0, v_1) = \beta(x_0, v_1'')$. From (2.4) we have

$$\begin{aligned}(2.5)\qquad \langle\langle \beta(y_1, n_1), \beta(y_2, n_2)\rangle\rangle &= \langle\langle \beta(x_0, v_1''), \beta(y_2, n_2)\rangle\rangle \\ &= T(x_0, y_2, v_1'', n_2) = 0\end{aligned}$$

because $\langle v_1'', In_2\rangle = \langle v_1'', Jn_2\rangle = \langle v_1'', Kn_2\rangle = 0$. Therefore, W_1 consists entirely of null vectors.

Let $\{\xi_i \oplus \eta_i \oplus \nu_i \oplus \mu_i, 1 \leq i \leq s\}$ be a basis of W_1 ($s = \dim W_1$). Then $\{\xi_i, 1 \leq i \leq s\}$ is a family of linearly independent vectors in $T_p^\perp M$, otherwise there exists a linear combination $\sum_{i=1}^s a_i\xi_i = 0$ with some $a_i \neq 0$, so

$$(2.6)\qquad \sum_{i=1}^{s} a_i(\xi_i \oplus \eta_i \oplus \nu_i \oplus \mu_i) = 0 \oplus \eta \oplus \nu \oplus \mu,$$

which combined with (2.5) implies $\eta = \nu = \mu = 0$, therefore (2.6) gives a nontrivial linear combination of $\{\xi_i \oplus \eta_i \oplus \nu_i \oplus \mu_i, 1 \leq i \leq s\}$.

We consider the positive definite inner product

$$(2.7)\quad g(\lambda_1 \oplus \lambda_2 \oplus \lambda_3, \rho_1 \oplus \rho_2 \oplus \rho_3) = \tfrac{1}{3}\{\langle\lambda_1, \rho_1\rangle + \langle\lambda_2, \rho_2\rangle + \langle\lambda_3, \rho_3\rangle\}$$

on $H_p = T_p^\perp M \oplus T_p^\perp M \oplus T_p^\perp M$.

Since we may suppose $\{\xi_i, 1 \leq i \leq s\}$ is a family of orthonormal vectors, by using (2.5) we get

$$\begin{aligned}&\langle \xi_i, \xi_j\rangle - g(\eta_i \oplus \nu_i \oplus \mu_i, \eta_j \oplus \nu_j \oplus \mu_j) \\ &\qquad = \tfrac{1}{3}\langle\langle \xi_i \oplus \eta_i \oplus \nu_i \oplus \mu_i, \xi_j \oplus \eta_j \oplus \nu_j \oplus \mu_j\rangle\rangle = 0,\end{aligned}$$

which implies $\{\eta_i \oplus \nu_i \oplus \mu_i, 1 \leq i \leq s\}$ is a family of orthonormal vectors in (H_p, g). Consequently, there exists an orthonormal transformation F: $T_p^\perp M \to H_p$ such that $F(\xi_i) = \eta_i \oplus \nu_i \oplus \mu_i$, $1 \leq i \leq s$. In particular, if $(x, n) \in T_pM \times N_p$, then $\beta(x, n) \in W_1$, therefore,

$$(2.8)\qquad F(\sigma(x, n)) = \sigma(x, In) \oplus \sigma(x, Jn) \oplus \sigma(x, Kn).$$

From (i) we know $In, Jn, Kn \in N_p$ for all $n \in N_p$, so

$$\begin{aligned}\sigma(In, In) \oplus \sigma(In, Jn) \oplus \sigma(In, Kn) &= F(\sigma(In, n)) = F(\sigma(n, In)) \\ &= \sigma(n, I^2n) \oplus \sigma(n, JIn) \oplus \sigma(n, KIn),\end{aligned}$$

which implies $\sigma(In, In) = -\sigma(n, n)$ and, similarly, $\sigma(Jn, Jn) = \sigma(Kn, Kn) = -\sigma(n, n)$. Now clearly, $\sigma(n, n) = 0$ for all $n \in N_p$, which proves (iii).

3. Proof of Main Theorem. We may suppose $n \geqslant 2$ and M is an Einstein space. According to [**6**] we have

$$A(x, y) = -\frac{\tau}{n+2}\langle Ix, y\rangle, \quad B(x, y) = -\frac{\tau}{n+2}\langle Jx, y\rangle,$$
$$C(x, y) = -\frac{\tau}{n+2}\langle Kx, y\rangle$$

for any two vectors $x, y \in T_pM$, where $4n\tau$ is the scalar curvature of M.

Let n be any unit vector in N_p. Then from the Lemma, Gauss equation and (2.1), we get $A(n, In) = 0$ so $\tau = 0$. This proves M is Ricci flat.

If S denotes the Ricci tensor of M, then for any $n \in N_p$, we have

$$0 = S(n, n) = -\sum_{i=1}^{4n} \|\sigma(n, e_i)\|^2, \tag{3.1}$$

where $\{e_i, 1 \leqslant i \leqslant 4n\}$ is an orthonormal basis of T_pM. Therefore, we get

$$\sigma(n, x) = 0 \quad \text{for all } n \in N_p \text{ and } x \in T_pM. \tag{3.2}$$

On the other hand, let RN_p be the relative nullity space at $p \in M$ and $v_p = \dim RN_p$. From (3.2), $N_p \subseteq RN_p$ so $v_p \geqslant 4(n-d)$. Let $v_0 = \min\{v_q/q \in M\}$ and $G = \{q \in M/v_q = v_0\}$. As is well known, G is an open subset of M on which RN_p $(p \in G)$ defines an involutive distribution whose leaves are complete v_0-planes.

Let L be the leaf through $p \in G$. Then since RN_p contains a quaternionic subspace of dimension $4m \geqslant 4(n-d)$, L contains Q^m, which is immersed by ϕ identically onto E^{4m}. Since M is Ricci flat, from [**1**, Theorem 2], we see that M is a Riemannian product of Q^m and a $4(n-m)$-dimensional Riemannian manifold P.

From [**3**] V is locally parallelizable, i.e., at each coordinate neighborhood U there exists local basis $\{I, J, K\}$ of V satisfying $\nabla I = \nabla J = \nabla K = 0$. Hence P is a Ricci flat quaternion Kaehlerian manifold with the induced quaternionic structure.

Since the second fundamental form of ϕ satisfies $\sigma(x, y) = 0$ for x tangent to Q^m and y tangent to P, [**4**] implies ϕsplits into a product of the identity map of Q^m onto E^{4m} and an isometric immersion of P into $E^{4(n-m)+d}$.

In particular, when $d = 1$, we get the following characterization of Q^n with its standard quaternionic structure.

COROLLARY. Q^n $(n > 1)$ *is the only complete, simply connected quaternion Kaehler manifold which can be isometrically immersed as a hypersurface of* E^{4n+1}.

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DEPARTAMENTO DE GEOMETRIA Y TOPOLOGIA, UNIVERSIDAD DE GRANADA, GRANADA, SPAIN