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POSITIVE MARTINGALES AND THEIR INDUCED MEASURES

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ABSTRACT. Conditions for the absolute continuity of probability measures are given in terms of limits of sequences of Radon-Nikodym derivatives. In the other direction, conditions for the optional stopping theorem for positive martingales are given in terms of properties of their induced measures.

Introduction. Let $(\Omega, \mathcal{E}, P)$ and $(\Omega, \mathcal{E}, Q)$ be probability spaces and let $(\mathcal{F}_n, n \in N)$ be an increasing sequence of sub-σ-fields of $\mathcal{E}$, where N is the set of nonnegative integers. If Q is absolutely continuous with respect to P over $\mathcal{F}_n$ we write $Q_n \ll P_n$ and $dQ_n = f_n dP_n$, where f_n is the Radon-Nikodym derivative over $\mathcal{F}_n$. $(f_n, n \in N)$ is then a positive martingale with respect to $(\mathcal{F}_n, n \in N)$ and P. Hence $\lim f_n$ exists a.s. P. Conversely, if $(f_n, n \in N)$ is a positive martingale with respect to $(\mathcal{F}_n, n \in N)$ and P, then there is a well-defined finitely additive set function Q on $\bigcup_{n=0}^{\infty} \mathcal{F}_n$ with $Q(A) = \int_A f_n\, dP$ for A in $\mathcal{F}_n$. Q is called the measure induced by $(f_n, n \in N)$. Set $\mathcal{F}_\infty = \sigma(\bigcup_{n=0}^{\infty} \mathcal{F}_n)$.

The following theorem brings together several well-known results on the convergence of positive martingales (see Ash [**1972**, §7.6] and Neveu [**1975**, Proposition III-1-1]) and indicates the general significance of the problem considered in this paper. All positive martingales in this paper are normalized.

GENERAL THEOREM. *Assume f_n is a positive martingale with respect to $(\mathcal{F}_n, n \in N)$ and P, and Q is the induced measure. Then the following are equivalent*:

(i) $\int \lim f_n\, dP = 1$.

(ii) $f_n \to \lim f_n$ *in* $L^1(P)$.

(iii) $(f_n, n \in N)$ *is uniformly integrable* (P).

(iv) *There exists an f with* $\mathcal{E}_p(f \mid F_n) = f_n$ *for n in N*.

(v) $f_k = \mathcal{E}_p(\lim f_n \mid \mathcal{F}_k)$ *for k in N*.

(vi) *Q extends to a σ-additive measure Q_∞ on $\mathcal{F}_\infty$ and $Q_\infty \ll P_\infty$*.

(vii) *Q extends to Q_∞ and* $dQ_\infty = \lim f_n\, dP_\infty$.

In a recent paper Kabanov, Lipcer and Shiryaev (hereafter K.L.S.) have given another necessary and sufficient condition for $Q_\infty \ll P_\infty$ and have shown its usefulness. The result (K.L.S. [**1977**, Lemma 6]) states that $Q_\infty \ll P_\infty$ if and only if $Q(\lim f_n = \infty) = 0$. They also show that two probability measures Q and P with $Q_n \ll P_n$ for n in N are singular over $\mathcal{F}_\infty$ if and only if $Q(\lim f_n = \infty) = 1$.

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§1 contains a simple proof of an extension of the results of K.L.S. The proof avoids the technical problems associated with dividing by zero in the K.L.S. approach. §2 gives some properties of likelihood ratios or extended Radon-Nikodym derivatives. §3 includes conditions for the absolute continuity and singularity over $\mathcal{F}_\nu$, the σ-field of events prior to a stopping time ν. The proofs show how useful the likelihood ratio can be. Moreover, the results lead to necessary and sufficient conditions for the optional stopping theorem of the same type as the K.L.S. conditions for absolute continuity.

1. Absolute continuity over $\mathcal{F}_\infty$. Let $R = \frac{1}{2}(P + Q)$. Then $Q \ll R$ over a general σ-field $\mathcal{F} \subset \mathcal{E}$. If $dQ = y\,dR$ then $Q(A) = \frac{1}{2}\int_A y\,dQ + \frac{1}{2}\int_A y\,dP$, so $\int_A (2 - y)\,dQ = \int_A y\,dP$ for all A in $\mathcal{F}$. In particular, by first letting $A = \{y = 2\}$ and then $A = \{y = 0\}$, we see that $Q(y = 0) = P(y = 2) = 0$.

LEMMA 1. (a) *$Q \ll P$ if and only if $Q(y = 2) = 0$.*
(b) *The following are equivalent*: (i) $Q \perp P$, (ii) $Q(y = 2) = 1$, (iii) $P(y = 0) = 1$.

PROOF. (a) Assume $Q(y = 2) = 0$ and $P(A) = 0$. Then $\int_A (2 - y)\,dQ = \int_A y\,dP = 0$. But $Q(2 - y > 0) = 1$. Hence $Q(A) = 0$. This shows $Q \ll P$.
Conversely, assume $Q \ll P$. Then since $P(y = 2) = 0$ we must have $Q(y = 2) = 0$.
(b) (i) $\Leftrightarrow$ (ii). Assume $Q(y = 2) = 1$. Then since $P(y = 2) = 0$ we have $Q \perp P$.
Conversely, let $P(A) = 0$ and $Q(A) = 1$ for some A. Then $\int_A (2 - y)\,dQ = \int_A y\,dP = 0$. Hence $Q(2 - y \neq 0) = 0$. That is, $Q(y = 2) = 1$.
The proof of (i) $\Leftrightarrow$ (iii) is similar. $\square$
An extension of the K.L.S. results is an immediate consequence.

THEOREM 1. *Assume $dQ_n = f_n\,dP_n$ and Q extends to a σ-additive measure on $\mathcal{F}_\infty$.*
(a) *The following are equivalent*: (i) $Q_\infty \ll P_\infty$, (ii) $Q(\lim f_n = \infty) = 0$, (iii) $Q(\limsup f_n = \infty) = 0$.
(b) *The following are equivalent*: (i) $Q_\infty \perp P_\infty$, (ii) $Q(\lim f_n = \infty) = 1$, (iii) $Q(\limsup f_n = \infty) = 1$, (iv) $P(\lim f_n = 0) = 1$.

PROOF. $dQ_n = f_n\,dP_n$ and $dR_n = \frac{1}{2}(1 + f_n)\,dP_n$. Thus $dQ_n = 2f_n(1 + f_n)^{-1}\,dR_n$, and and by the General Theorem $2f_n(1 + f_n)^{-1}$ converges a.s. R to y_∞, where $dQ_\infty = y_\infty\,dR_\infty$. In particular, $\lim 2f_n(1 + f_n)^{-1} = y_\infty$ a.s. Q.
(a) and (b) (i) $\Leftrightarrow$ (ii). $Q(y_\infty = 2) = Q(\lim 2f_n(1 + f_n)^{-1} = 2) = Q(\lim f_n = \infty)$. Now apply Lemma 1.
(b) (ii) $\Leftrightarrow$ (iii). $Q(\lim 2f_n(1 + f_n)^{-1}$ exists$) = 1$.
(i) $\Leftrightarrow$ (iv). $P(y_\infty = 0) = P(\lim 2f_n(1 + f_n)^{-1} = 0) = P(\lim f_n = 0)$. Now apply Lemma 1. $\square$
The condition that Q extend to a σ-additive measure on $\mathcal{F}_\infty$ is very mild. It is always the case in the standard situation where $\Omega = R^\infty$, $(X_n, n \in N)$ are the coordinate functions, $\mathcal{F}_n = \sigma(X_0, \ldots, X_n)$, and f_n is a function of $(X_1, \ldots, X_n)$. This follows from the Kolmogoroff existence theorem. In this case we say $(f_n, n \in N)$ is a standard positive martingale.

2. Likelihood ratios. The Lebesgue decomposition also follows from the identity $\int_A (2-y)\,dQ = \int_A y\,dP$. Let $h = y(2-y)^{-1}$, where this expression is defined as ∞ for $y = 2$.

LEMMA 2. (a) $Q(A) = \int_A h\,dP + Q(A \cap \{h = \infty\})$, *where* $P(h = \infty) = 0$.
(b) *The following are equivalent*: (i) $Q \ll P$, (ii) $Q(h = \infty) = 0$, (iii) $\int h\,dP = 1$.
(c) *The following are equivalent*: (i) $Q \perp P$, (ii) $Q(h = \infty) = 1$, (iii) $\int h\,dP = 0$.

PROOF. (a)

$$Q(A \cap \{h \neq \infty\}) = \int_{A \cap \{y \neq 2\}} \frac{2-y}{2-y}\,dQ = \int_{A \cap \{y \neq 2\}} \frac{y}{2-y}\,dP = \int_A h\,dP$$

since $\{y = 2\} = \{h = \infty\}$ and $P(y = 2) = 0$ always.
(b) and (c) (i) $\Leftrightarrow$ (ii). From Lemma 1.
(b) and (c) (ii) $\Leftrightarrow$ (iii). Since $1 = Q(\Omega) = \int h\,dP + Q(h = \infty)$. $\square$
h is an extended Radon-Nikodym derivative called the likelihood ratio in statistics. Let h_n be the likelihood ratio over $\mathfrak{F}_n$. We have

LEMMA 3. $h_n \to h_\infty$ *a.s.* $P + Q$.

PROOF. Let $dQ_n = y_n\,dR_n$. Then $y_n \to y_\infty$ a.s. R. Hence $h_n = y_n(2 - y_n)^{-1} \to y_\infty(2 - y_\infty)^{-1} = h_\infty$ a.s. R. $\square$
This lemma extends K.L.S. [**1977**, Lemma 5], which assumes $Q_n \ll P_n$ for n in N. It shows that in a certain sense likelihood ratios are easier to work with than Radon-Nikodym derivatives since limits of likelihood ratios are always likelihood ratios.

COROLLARY. *Let* P_∞ *and* Q_∞ *be probability measures on* $\mathfrak{F}_\infty$. $Q_\infty \perp P_\infty$ *if and only if* $Q(\lim h_n = \infty) = 1$.

PROOF. $Q_\infty \perp P_\infty \Leftrightarrow Q(h_\infty = \infty) = 1 \Leftrightarrow Q(\lim h_n = \infty) = 1$. $\square$
This corollary does not require $Q_n \ll P_n$ for n in N as in the previous results.

3. Absolute continuity over $\mathfrak{F}_\nu$. Let $\overline{N} = N \cup \{\infty\}$. ν is called a stopping time with respect to $\{\mathfrak{F}_n, n \in \overline{N}\}$ if $\{\nu = n\} \in \mathfrak{F}_n$ for n in $\overline{N}$. $\mathfrak{F}_\nu = \{A \mid A \cap \{\nu = n\} \in \mathfrak{F}_n$ for n in $\overline{N}\}$ is the σ-field of events up to time ν. Over the σ-field $\mathfrak{F}_\nu$ we write $dQ_\nu = y_\nu\,dR_\nu$.

LEMMA 4. *Assume P and Q are probability measures on* $\mathfrak{F}_\infty$.
(a) $y_\nu = y_n$ *on* $\{\nu = n\}$ *for n in* $\overline{N}$.
(b) $h_\nu = h_n$ *on* $\{\nu = n\}$ *for n in* $\overline{N}$.

PROOF. (a) If A is in $\mathfrak{F}_\nu$ then $A = \bigcup_{n \in \overline{N}}(A_n \cap \{\nu = n\})$ for A_n in $\mathfrak{F}_n$. But $Q(A_n \cap \{\nu = n\}) = \int_{A_n \cap \{\nu = n\}} y_n\,dR$ since $A_n \cap \{\nu = n\} \in \mathfrak{F}_n$. Thus $y_\nu = y_n$ on $\{\nu = n\}$ by the uniqueness of the Radon-Nikodym derivative.
(b) $h_\nu = y_\nu(2 - y_\nu)^{-1}$. $\square$

COROLLARY. (a) *The following are equivalent*:
(i) $Q_\nu \ll P_\nu$.
(ii) $Q(h_\nu = \infty) = 0$.
(iii) $Q(\nu = n, h_n = \infty) = 0$ *for n in* $\overline{N}$.

(iv) $\int h_\nu \, dP = 1$.

(b) *The following are equivalent*:

(i) $Q_\nu \perp P_\nu$.

(ii) $Q(h_\nu = \infty) = 1$.

(iii) $Q(\nu = n, h_n < \infty) = 0$ *for* n *in* $\overline{N}$.

(iv) $\int h_\nu \, dP = 0$.

PROOF. (a) and (b) (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iv) follows by Lemma 2, and (ii) $\Leftrightarrow$ (iii) follows from Lemma 4. $\square$

THEOREM 2. *Assume* Q *and* P *are probability measures. Assume* $dQ_n = f_n \, dP_n$ *for* n *in* N. *Then*:

(a) $Q_\nu \ll P_\nu$ *if and only if* $Q(\lim f_n = \infty, \nu = \infty) = 0$,

(b) $Q_\nu \perp P_\nu$ *if and only if* $Q(\lim f_n = \infty, \nu = \infty) = 1$.

PROOF. $h_n = f_n < \infty$ for n in N and $h_\infty = \lim f_n$. Now apply the Corollary to Lemma 4. $\square$

COROLLARY 1. $dQ_n = f_n \, dP_n$ *for* n *in* N *and* $Q(\nu = \infty) = 0$ *then* $Q_\nu \ll P_\nu$.

COROLLARY 2. *If* $dQ_n = f_n \, dP_n$ *for* n *in* N *and* $Q_\infty \perp P_\infty$, *then*:

(a) $Q_\nu \ll P_\nu$ *if and only if* $Q(\nu = \infty) = 0$,

(b) $Q_\nu \perp P_\nu$ *if and only if* $Q(\nu = \infty) = 1$.

PROOF. If $Q_\infty \perp P_\infty$ then $Q(\lim f_n = \infty) = 1$ by Theorem 1. $\square$

It also follows from the Lebesgue decomposition that when $Q_\nu \ll P_\nu$ we have $dQ_\nu = f_\nu \, dP$, where $f_\nu = f_n$ for n in N and $f_\nu = \lim f_n$ on $\{\nu = \infty\}$.

These results have an interesting interpretation in terms of the optional stopping problem. The next theorem gives necessary and sufficient conditions for two versions of the optional stopping theorem for positive martingales that are analogous to the K.L.S. necessary and sufficient conditions for absolute continuity.

THEOREM 3. *Assume* $(f_n, n$ *in* $N)$ *is a positive martingale and the induced measure* Q *is* σ*-additive. Then*:

(a) $1 = \int_{\{\nu<\infty\}} f_\nu \, dP + \int_{\{\nu=\infty\}} \lim f_n \, dP + Q(\nu = \infty, \lim f_n = \infty)$.

(b) $\int f_\nu \, dP = 1 \Leftrightarrow Q(\nu = \infty, \lim f_n = \infty) = 0$.

(c) *The following are equivalent*:

(i) $1 = \int_{\{\nu<\infty\}} f_\nu \, dP$,

(ii) $Q(\nu = \infty, \lim f_n = \infty) = 0$ *and* $P(\lim f_n > 0, \nu = \infty) = 0$,

(iii) $Q(\nu = \infty) = 0$.

PROOF. (a) follows from the Lebesgue decomposition in Lemma 2 since $h_\nu = f_\nu$ and $\{h_\nu = \infty\} = \{\lim f_n = \infty, \nu = \infty\}$.

(b) and (c) (i) $\Leftrightarrow$ (ii) follows from (a).

(c) (ii) $\Leftrightarrow$ (iii) again follows from Lemma 2 since

$$Q(\nu = \infty) = \int_{\{\nu=\infty\}} f_\nu \, dP + Q(\nu = \infty, f_\nu = \infty). \quad \square$$

The condition that Q is σ-additive can, of course, be replaced by the condition that $(f_n, n \text{ in } N)$ is a standard positive martingale.

In the general theory of martingales ν is called a regular stopping time if $f_{\nu \wedge n}$ converges to f_ν in $L^1(P)$. It follows from Scheffe's lemma that this is the case if $\int f_\nu \, dP = 1$. Thus Theorem 3 also gives necessary and sufficient conditions for ν to be regular.

The final corollary, which is an immediate consequence of the theorem, shows the usefulness of the conditions in terms of Q.

COROLLARY. *Under the conditions of Theorem* 3, *let* ν_1 *and* ν_2 *be stopping times with* $\{\nu_1 = \infty\} \subset \{\nu_2 = \infty\}$. *Then*:

(a) $\int f_{\nu_2} \, dP = 1 \Rightarrow \int f_{\nu_1} \, dP = 1$,

(b) $\int_{\{\nu_2 < \infty\}} f_{\nu_2} \, dP = 1 \Rightarrow \int_{\{\nu_1 < \infty\}} f_{\nu_1} \, dP = 1$.

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