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DIFFERENTIABILITY CRITERIA AND HARMONIC FUNCTIONS ON B^n

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ABSTRACT. Let B^n be the unit ball in $\mathbf{F}^n$ where $\mathbf{F}$ is either $\mathbf{R}, \mathbf{C}$ or $\mathbf{H}$. The space B^n is a classical rank one symmetric space with respect to the action of a Lie group G. Suppose f is a G-harmonic function on B^n all of whose derivatives of order $\leqslant k$ are bounded. Our main result obtains restrictions on f depending on k.

1. Introduction. Let $\mathbf{F}$ denote one of the classical fields $\mathbf{R}, \mathbf{C}$ or $\mathbf{H}$. Consider $\mathbf{F}^{n+1}$ as a right vector space over $\mathbf{F}$, and on $\mathbf{F}^{n+1}$ define the quadratic form $Q(x_1,\ldots,x_{n+1}) = |x_1|^2 + \cdots + |x_n|^2 - |x_{n+1}|^2$. If G is the connected component of the group of all $\mathbf{F}$-linear transformations of $\mathbf{F}^{n+1}$ with determinant one which preserve Q, G is $\mathrm{SO}^0(n,1)$, $\mathrm{SU}(n,1)$ or $\mathrm{Sp}(n,1)$ according as $\mathbf{F}$ is $\mathbf{R}, \mathbf{C}$ or $\mathbf{H}$. If

$$B^n = \left\{\vec{x} \in \mathbf{F}^n\colon |x_1|^2 + \cdots + |x_n|^2 < 1\right\},$$

G acts transitively on B^n making B^n a Riemannian symmetric space. If $g \in G$ we have

$$g = \begin{pmatrix} A & \vec{b} \\ \vec{c}^t & d \end{pmatrix}$$

with A an $n \times n$ matrix, $\vec{b}$ and $\vec{c}$ in $\mathbf{F}^n$, and $d \in \mathbf{F}$. For $\vec{x} \in B^n$ and g as above,

$$g(\vec{x}) = (A\vec{x} + \vec{b})(d + \langle \vec{x}, \vec{c}\rangle)^{-1},$$

where $\langle \vec{x}, \vec{c}\rangle = x_1c_1 + \cdots + x_nc_n$. Let K denote the subgroup of G fixing $\vec{0}$.

For k a nonnegative integer let $\mathfrak{D}_k(\mathbf{F}^n)$ denote the set of all constant coefficient differential operators on $\mathbf{F}^n$ of order $\leqslant k$.

DEFINITION 1.1. Let k be a nonnegative integer. We say that $F \in C_k(B^n)$ if for every $D \in \mathfrak{D}_k(\mathbf{F}^n)$, DF is uniformly continuous on B^n (or equivalently DF extends to a continuous function on $\overline{B^n}$). We say that $F \in C_\infty(B^n)$ if $F \in C_k(B^n)$ for all k.

If Δ is the G-invariant Laplace operator on B^n, let $\mathcal{H}(B^n) = \{F \in C^\infty(B^n)\colon \Delta F = 0\}$, and for k a nonnegative integer or ∞ let $\mathcal{H}_k(B^n) = \mathcal{H}(B^n) \cap C_k(B^n)$.

Our main result gives a precise description of the representations of K which can occur in $\mathcal{H}_k(B^n)$. This is accomplished by using known results on principal series representations [**JW**].

For the case $\mathbf{F} = \mathbf{C}$ our result may be obtained from the explicit Fourier expansions for functions in $\mathcal{H}(B^n)$ which were given by Folland [**F**]. In fact, more

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can be obtained as was shown to us by Walter Rudin. Our proofs follow from results in representation theory, and this in itself makes the proofs somewhat novel.

2. Topological preliminaries. Let k be a nonnegative integer and F a function on B^n. For $D \in \mathfrak{D}_k(\mathbf{F}^n)$ set

$$\nu_D(F) = \sup_{\vec{x} \in B^n} |DF|.$$

The norms ν_D turn $\mathcal{H}_k(B^n)$ into a Banach space and $\mathcal{H}_\infty(B^n)$ into a Fréchet space.

Fix k a nonnegative integer or ∞ and set $V = \mathcal{H}_k(B^n)$. For $g \in G$ and $f \in V$ set $L_g f(\vec{x}) = f(g^{-1}(\vec{x}))$. Now $L_g f \in V$ and thus we have a representation of G on V.

LEMMA 2.1. *The map $G \times V \to V$, given by $(g, f) \to L_g f$, is continuous.*

PROOF. The topology of G is induced from the norm $\|g\| = \sup|g_{ij}|$. If D is any homogeneous differential operator with constant coefficients of order $r \geqslant 2$ we have

$$\lim_{g \to I} \left(\sup_{x \in B^n} \|D(g^{-1}\vec{x})\| \right) = 0.$$

Moreover, if $\vec{v} \in \mathbf{F}^n$ set

$$D_{\vec{v}}(g^{-1}\vec{x}) = \frac{d}{dt} g^{-1}(\vec{x} + t\vec{v}) \,|_{t=0}.$$

Then we have

$$\lim_{g \to I} \left(\sup_{\vec{x} \in B^n} \|D_{\vec{v}}(g^{-1}\vec{x}) - \vec{v}\| \right) = 0.$$

Our result now follows from repeated application of the chain rule and the limits above.

Let $\hat{K}$ be the unitary dual of K, and for $\delta \in \hat{K}$ set

$$E_\delta = d(\delta) \int_K \chi_\delta(k) L_k \, dk.$$

From Lemma 4 of **[H-C]** we have $V_F = \sum_{\delta \in \hat{K}} E_\delta V$ is dense in V. If $\mathfrak{g}$ is the Lie algebra of G and $\mathfrak{U}(\mathfrak{g})$ is the enveloping algebra of $\mathfrak{g}$, we obtain a representation of $\mathfrak{U}(\mathfrak{g})$ on V_F. This representation is a principal series representation. Now using results of **[JW]**, we determine for which $\delta \in \hat{K}$ is $V_\delta = E_\delta V \neq \{0\}$.

3. Some results from representation theory. Let M be the subgroup of K which fixes the point $\vec{e}_1$ in ∂B^n. Let $\hat{K}_0$ be the set of equivalence classes of representations in $\hat{K}$ which have an M-fixed vector. Then, if V is as in §2,

$$V_F = \sum_{\delta \in \hat{K}_0} V_\delta,$$

and if $V_\delta \neq \{0\}$, K acts irreducibly on V_δ, and V_δ contains only one M-fixed vector (see **[JW]**).

(I) If $\mathbf{F} = \mathbf{R}$, $K = \mathrm{SO}(n)$ $(n \geqslant 3)$, and $\hat{K}_0$ may be identified with the nonnegative integers. If $p \in \hat{K}_0$ the corresponding representation of K is the action of K on $H^p(\mathbf{R}^n)$, the space of homogeneous harmonic polynomials on $\mathbf{R}^n$ of degree p.

(II) If $\mathbf{F} = \mathbf{C}$, $K = \mathrm{S}(\mathrm{U}(n) \times \mathrm{U}(1))$ $(n \geqslant 2)$, and $\hat{K}_0$ may be identified with the set of pairs of nonnegative integers. If $(p, q) \in \hat{K}_0$, the corresponding representation of K is the action of K on $H^{p,q}(\mathbf{C}^n)$, the space of homogeneous harmonic polynomials f on $\mathbf{C}^n$ of degree $p + q$ for which $f(\lambda\vec{x}) = \lambda^p\bar{\lambda}^q f(\vec{x})$ for $\lambda \in \mathbf{C}$.

(III) If $\mathbf{F} = \mathbf{H}$, $K = \mathrm{Sp}(n) \times \mathrm{Sp}(1)$, and $\hat{K}_0$ may be identified with the set of pairs of nonnegative integers (m, l) such that $m - l$ is a nonnegative even integer. For a description of the representation of K on $H^{m,l}(\mathbf{H}^n)$ corresponding to (m, l), see [**JW**].

Clearly, the restriction of any linear functions on $\mathbf{F}^n$ to B^n is in V_F. Let L be the smallest $\mathfrak{U}(\mathfrak{g})$-invariant subspace of V_F which contains the restrictions of all linear functions. From Theorem 5.1 of [**JW**] we obtain

THEOREM 3.1. (i) *In all cases either $V_F = L$ or the action of $\mathfrak{U}(\mathfrak{g})$ on V_F/L is irreducible, and if $L \subsetneq V_F$, $V_\delta \neq \{0\}$ for any $\delta \in \hat{K}_0$.*

(ii) *If $\mathbf{F} = \mathbf{R}$, $L = V_F$ and $V_r \neq \{0\}$ for any r.*

(iii) *If $\mathbf{F} = \mathbf{C}$, $L = \Sigma(V_{(p,0)} + V_{(0,q)})$ and $V_{(p,0)} \neq \{0\}$ for any p and $V_{(0,q)} \neq \{0\}$ for any q.*

(iv) *If $\mathbf{F} = \mathbf{H}$, $L = \Sigma_{m-l \leqslant 2} V_{(m,l)}$ and $V_{(m,l)} \neq \{0\}$ for any (m, l) with $m - l \leqslant 2$.*

COROLLARY. (i) *If $\mathbf{F} = \mathbf{C}$, $V_{(p,q)} \neq \{0\}$ for every (p, q) if and only if $V_{(1,1)} \neq \{0\}$.*

(ii) *If $\mathbf{F} = \mathbf{H}$, $V_{(m,l)} = \{0\}$ for every (m, l) if and only if $V_{(4,0)} \neq \{0\}$.*

PROOF. The proof of both statements follows from Theorem 3.1(i) and the fact that if $\mathbf{F} = \mathbf{C}$, $V_{(1,1)} \subsetneq L$ and if $\mathbf{F} = \mathbf{H}$, $V_{(4,0)} \subsetneq L$.

4. Our main result. (I) If $\mathbf{F} = \mathbf{C}$ a simple calculation shows that the function $\vec{x} \to |x_1|^2 - \|\vec{x}\|^2/n$ is in $H^{1,1}(\mathbf{C}^n)$. If V is as in §3, from [**F**] we see that $V_{(1,1)} \neq \{0\}$ if and only if $V_{(1,1)}$ contains the harmonic function

$$F(1, 1; n + 2; \|\vec{x}\|^2)\left(|x_1|^2 - \|\vec{x}\|^2/n\right).$$

(II) If $\mathbf{F} = \mathbf{H}$ a simple calculation shows that the function

$$\vec{x} \to |x_1|^4 F\left(-2, -3; 2(n-1); \frac{|x_1|^2 - \|\vec{x}\|^2}{|x_1|^2}\right) = u(\vec{x})$$

is in $H^{4,0}(\mathbf{H}^n)$, and calculating with the Casimir operator of $\mathrm{Sp}(n, 1)$ we see that $F(2, 1; 2n + 4; \|\vec{x}\|^2)u(\vec{x})$ is harmonic and must be in $V_{(4,0)}$ if $V_{(4,0)} \neq \{0\}$.

If $c - a - b > 0$ we have

$$\lim_{t \to 1} F(a, b; c; t) = \frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)}$$

[**WW**, p. 282] while if $c - a - b = 0$ the limit does not exist. As

$$\frac{d}{dt} F(a, b; c; t) = \frac{ab}{c} F(a + 1, b + 1; c + 1; t)$$

we now obtain the following.

THEOREM 4.1. *Let k be a nonnegative integer or ∞ and let $V = \mathcal{H}_k(B^n)$. Then the following are true.*

(i) *If* $\mathbf{F} = \mathbf{R}$, $V_p \neq \{0\}$ *for any* p.

(ii) *If* $\mathbf{F} = \mathbf{C}$ *and* $k \leqslant n - 1$, $V_{(p,q)} \neq \{0\}$ *for any* (p, q), *while if* $k \geqslant n$, $V_F = L$.

(iii) *If* $\mathbf{F} = \mathbf{H}$ *and* $k \leqslant 2n$, $V_{(m,l)} \neq \{0\}$ *for any* (m, l), *while if* $k \geqslant 2n + 1$, $V_F = L$.

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