

## G-SPACES WITH PRESCRIBED EQUIVARIANT COHOMOLOGY

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**ABSTRACT.** Let  $G$  be a finite group. In this note we study the question of realizing a collection of graded commutative algebras over  $\mathbf{Q}$  as the cohomology algebras with rational coefficients of the fixed point sets  $X^H$  ( $H < G$ ) of a  $G$ -space  $X$ .

Let  $\mathcal{A}$  be a graded commutative algebra over  $\mathbf{Q}$ . The question of realizing  $\mathcal{A}$  as the cohomology with rational coefficients of a space  $X$  is answered by Quillen [4] and more directly by Sullivan [5]. In particular, Sullivan constructs a space  $X$  of finite type, i.e.  $\pi_i(X)$  is a finitely generated abelian group for every  $i$ , which realizes  $\mathcal{A}$ . Now let  $G$  be a finite group which acts on  $\mathcal{A}$  from the left by algebra isomorphisms. Because of the functoriality of the constructions in [4] and [5] one can construct a  $G$ -space  $X$  such that  $H^*(X; \mathbf{Q}) \cong \mathcal{A}$ , where the isomorphism is  $G$ -equivariant. The space  $X$  in both cases is a rational space.

In this note we consider a more general question. Let  $\mathcal{C}_G$  be the category of canonical orbits of a finite group  $G$  [1]. The objects of  $\mathcal{C}_G$  are the quotient spaces  $G/H$ , where  $H$  is a subgroup of  $G$  ( $H < G$ ), and the morphisms are the  $G$ -maps between them, where  $G$  acts on  $G/H$  by left multiplication.

**DEFINITION 1.** A system of graded commutative algebras (GA's) for  $G$  is a covariant functor from  $\mathcal{C}_G$  into the category of graded commutative connected algebras over  $\mathbf{Q}$ .

We recall that a GA  $\mathcal{A}$  is said to be connected if  $\mathcal{A}^0 = \mathbf{Q}$  and is said to be of finite type if  $\mathcal{A}^n$  is a finite-dimensional vector space over  $\mathbf{Q}$ .

Let  $X$  be a  $G$ -space such that each fixed point set  $X^H$ ,  $H < G$ , is nonempty and connected. Given  $X$ , a system of GA's  $\underline{H}^*(X)$  is defined by

$$\underline{H}^*(X)(G/H) \cong H^*(X^H; \mathbf{Q})$$

on objects of  $\mathcal{C}_G$ . If  $f: G/H \rightarrow G/K$  is a  $G$ -map, then there exists an element  $g \in G$  such that  $g^{-1}Hg < K$ , and the map  $f$  is determined by  $H \mapsto gK$ . The map  $f$  induces a map  $\tilde{f}: X^K \rightarrow X^H$  by  $x \mapsto gx$  and therefore a unique map  $\underline{H}^*(X)(f) \equiv \tilde{f}^*: H^*(X^H; \mathbf{Q}) \rightarrow H^*(X^K; \mathbf{Q})$ .

The main result of this paper is the following

**THEOREM 2.** *Given a system  $\mathbf{H}$  of connected GA's of finite type, there exists a  $G$ -CW-complex  $X$  of finite type such that  $\underline{H}^*(X) \cong \mathbf{H}$ .*

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We say that a  $G$ -space  $X$  is of finite type if each  $X^H$  ( $H < G$ ) is of finite type.

For the proof of the theorem we need some additional terminology and the following lemmas.

DEFINITION 3. A coefficient (respectively rational coefficient) system for  $G$  is a contravariant functor from  $\mathcal{C}_G$  into the category of abelian groups (respectively the category of vector spaces over  $\mathbf{Q}$ ).

For instance, if  $X$  is a  $G$ -CW-complex [1], one can define coefficient systems  $\underline{\pi}_n(X)$  and  $\underline{C}_n(X)$  by

$$\underline{\pi}_n(X)(G/H) \equiv \pi_n(X^H) \quad \text{and} \quad \underline{C}_n(X)(G/H) \equiv C_n(X^H; \mathbf{Z}),$$

where the latter is the cellular complex of  $X^H$ .

DEFINITION 4. Let  $A$  be a coefficient system of finitely generated abelian groups and let  $M$  be a rational coefficient system. Let  $f: A \rightarrow M$  be a monomorphism. The system  $A$  is called a system of lattices for  $M$  if  $f(G/H) \otimes \text{id}$ :

$$A(G/H) \otimes \mathbf{Q} \xrightarrow{\cong} M(G/H) \quad \text{for all subgroups } H < G.$$

LEMMA 5. Every rational coefficient system  $M$  has a system of lattices  $A$ .

PROOF. The construction of  $A$  is done by induction on the subgroups of  $G$ . We first choose a lattice for the vector space  $M(G/G)$ .

Let  $H < G$  be a subgroup and assume inductively that  $A(G/K)$  is constructed for all subgroups  $K < G$  which contain  $H$  properly.  $M(G/H)$  is a  $\mathbf{Q}(NH/H)$ -module and there exists a  $\mathbf{Z}(NH/H)$ -submodule  $B$  such that  $B \otimes_{\mathbf{Z}} \mathbf{Q} \cong M(G/H)$  (cf. [2, p. 495]). We consider bases for  $B$  and  $A(G/K_1), \dots, A(G/K_m)$  regarded as  $\mathbf{Z}$ -modules, where  $K_1, \dots, K_m$  are representatives of all conjugacy classes of subgroups of  $G$  which contain  $H$  as a maximal subgroup. Let  $A_1, \dots, A_m$  be the matrices which represent the linear maps  $M(G/K_i) \rightarrow M(G/H)$  (induced by the projections  $G/H \rightarrow G/K_i$ ) with respect to the bases above. Let  $N$  be an integer such that  $NA_1, \dots, NA_m$  are matrices with integral entries. We multiply the basis elements of  $B$  by  $1/N$  and consider the new  $\mathbf{Z}$ -module generated this way. Obviously, it is a  $\mathbf{Z}(NH/H)$ -module and it has the property that it contains the images of  $A(G/K)$  ( $K > H$ ) via the maps induced by the projections. Hence, we have constructed  $A(G/H)$ . This completes the proof of the lemma.

LEMMA 6. Let  $A'$  be a coefficient system of finitely generated abelian groups and let  $M$  be a rational coefficient system. Let  $f': A' \rightarrow M$  be a monomorphism. Then  $M$  has a system of lattices  $A$  which extends  $A'$ , i.e. there exist monomorphisms  $i$  and  $f$  such that the diagram

$$\begin{array}{ccc} A' & \xrightarrow{f'} & M \\ i \searrow & & \nearrow f \\ & A & \end{array}$$

commutes.

PROOF. The same inductive argument works as above. In order to construct  $A(G/H) \supset A'(G/H)$  we use the semisimplicity of  $\mathbf{Q}(NH/H)$  to split off

$$A'(G/H) \otimes \mathbf{Q} \subset M(G/H).$$

Let  $\bar{M}$  be a  $\mathbf{Q}(NH/H)$ -submodule of  $M(G/H)$  which is a complement of  $A'(G/H) \otimes \mathbf{Q}$ , and let  $\bar{B}$  be a  $\mathbf{Z}(NH/H)$ -submodule of  $\bar{M}$  such that  $\bar{B} \otimes \mathbf{Q} = \bar{M}$ . We choose a basis for  $A'(G/H) \oplus \bar{B}$  and proceed as above.

PROOF OF THEOREM 2. We first use D. Sullivan's spatial realization of an algebra [5]. It is a functor  $F$  from the category of differential graded commutative algebras (DGA's) into the category of simplicial sets such that  $H^*(F(\mathcal{A}); \mathbf{Q}) \cong H^*(\mathcal{A})$  in a canonical way. Composing  $F$  with the geometric realization functor  $S \mapsto |S|$ , we get a functor into the category  $\mathfrak{S}$  of CW-complexes.

Let  $\underline{F}: \mathcal{C}_G \rightarrow \mathfrak{S}$  be the functor defined by  $\underline{F}(G/H) \equiv |F(\mathbf{H}(G/H))|$  on objects of  $\mathcal{C}_G$ , where  $\mathbf{H}(G/H)$  is considered as a DGA with differential equal to 0.

Such a functor is called an  $\mathcal{C}_G$ -space in [3]. Given an  $\mathcal{C}_G$ -space  $T$ , Elmendorf constructs a  $G$ -space  $C(T)$  in a functorial way such that  $C(T)^H$  is naturally homotopy equivalent to  $T(G/H)$ , for  $H < G$ . Let  $Y \equiv C(\underline{F})$ . By construction,  $Y$  is a  $G$ -CW-complex which realizes  $\mathbf{H}$ . Moreover,  $Y$  (and each  $Y^H$ ) is a rational space.

Next we construct a  $G$ -CW-complex  $X$  of finite type and a  $G$ -map  $f: X \rightarrow Y$  such that  $f^H: X^H \rightarrow Y^H$  is a rationalization for every subgroup  $H < G$ . We proceed by induction on the equivariant Postnikov decomposition of  $Y$  [6] starting from a point. Let  $X_n$  be a  $G$ -CW-complex of finite type and let

$$i_n: X_n \xrightarrow{i_n} Y_n$$

be a  $G$ -map which is a  $G$ -rationalization, i.e.

$$(i_n)_z: \pi_k(X_n) \otimes \mathbf{Q} \xrightarrow{\cong} \pi_k(Y_n).$$

Here  $Y_n$  is the  $n$ th stage of the equivariant Postnikov decomposition of  $Y$ . The  $(n + 1)$ st equivariant  $k$ -invariant of  $Y$  is an equivariant cohomology class represented by a natural transformation

$$k: \underline{C}_{n+2}(Y_n) \rightarrow \underline{\pi}_{n+1}(Y).$$

We compose  $k$  with  $i_n: \underline{C}_{n+2}(X_n) \rightarrow \underline{C}_{n+2}(Y_n)$  and consider the image of  $k \circ i_n$  in  $\underline{\pi}_{n+1}(Y)$ . By Lemma 6, we can extend it to a system of lattices  $A$  of  $\underline{\pi}_{n+1}(Y)$ . Let  $\hat{A}$  be any coefficient system of finitely generated abelian groups which surjects onto  $A$ ,

$$0 \rightarrow T \rightarrow \hat{A} \rightarrow A \rightarrow 0,$$

where  $T(G/H)$  is finite for every  $G/H$ . Consider the diagram:

$$\begin{array}{ccc} \underline{C}_{n+2}(Y_n) & \xrightarrow{k} & \underline{\pi}_{n+1}(Y) \supset A \\ i_n \uparrow & & \uparrow \\ \underline{C}_{n+2}(X_n) & \xrightarrow{k'} & \hat{A} \end{array}$$

Since  $\underline{C}_{n+2}(X_n)$  is projective in the category of coefficient systems [1], there is a lifting  $k'$ . Since  $k \circ i_n$  is a cocycle, it annihilates the cycles of  $\underline{C}_{n+2}(X_n)$ . Therefore,  $k'$  maps the cycles of  $\underline{C}_{n+2}(X_n)$  to torsion elements of  $\hat{A}$ . Hence a multiple  $m \cdot k'$  is a cocycle. Now, we construct  $X_{n+1}$  from  $X_n$  such that  $\underline{\pi}_{n+1}(X_{n+1}) \equiv \hat{A}$ , and with  $m \cdot k'$  as the  $(n+1)$ st  $k$ -invariant. Note that over the rationals,  $k$  and  $m \cdot k$  determine the same  $(n+1)$ st stage  $Y_{n+1}$  of the Postnikov tower. So  $Y_{n+1}$  is an equivariant rationalization of  $X_{n+1}$ .

In exactly the same way as in Theorem 2, we can prove the following statement.

**THEOREM 7.** *Given a system  $\mathfrak{C}$  of connected nilpotent differential graded algebras over  $\mathbf{Q}$  of finite type there exists a  $G$ -CW-complex  $X$  of finite type and a morphism  $\mathfrak{C} \rightarrow \underline{\mathfrak{S}}_X$  which induces an isomorphism on cohomology for every  $G/H$  ( $H < G$ ).*

The definition of a system of DGA's is analogous to Definition 1. A system of DGA's may not be injective in the sense of [6]. The system of DGA's  $\underline{\mathfrak{S}}_X$  was introduced and studied in [6]. By definition,  $\underline{\mathfrak{S}}_X(G/H) \equiv \underline{\mathfrak{S}}_{X^H}$  on objects, where  $\underline{\mathfrak{S}}_X$  is the Sullivan-De Rham complex of PL forms on  $X$  [5].

A DGA  $\mathfrak{C}$  is said to be connected if  $H^0(\mathfrak{C}) = \mathbf{Q}$  and it is said to be of finite type if  $H^i(\mathfrak{C})$  is a finite-dimensional vector space over  $\mathbf{Q}$ . For the definition of nilpotency see [5].

Theorem 7 generalizes a result by Sullivan [5] in the nonequivariant case (trivial  $G$ -action).

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