PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 89, Number 4, December 1983

NON-TYCHONOFF e-COMPACTIFIABLE SPACES

K. P. HART AND J. VERMEER

ABSTRACT. We construct a non-Tychonoff space X which is e-compactifiable, thus answering a question of S. Hechler. We also answer a question of R. M. Stephenson: whether there exists a Tychonoff space, the largest e-compactification of which has a noncompact semiregularization.

1. Introduction. All spaces are Hausdorff. In **[He]** S. Hechler introduced the class of e-compactifiable spaces, i.e. spaces which admit an e-compactification. He posed the question whether there exist non-Tychonoff e-compactifiable spaces. We show that such spaces exist. In **[St]** R. M. Stephenson observed that an e-compactifiable space has a largest e-compactification eX, and he asked whether the space $(eX)_S$—the semiregularization of eX—is always compact. We show that this need not be the case, even if the space X is assumed to be Tychonoff. The example of the space we present is based on an example of J. Chaber.

2. Preliminary definitions and theorems.

DEFINITION 2.1 **[He]**. Let D be a dense subspace of X. X is said to be *e-compact with respect to D* if each open cover of X contains a finite subcollection that covers D. If so, X is called an *e-compactification* of D and D is called *e-compactifiable*. $\square$

Observe that within this terminology the expression "let X be an e-compact space" is meaningless. From this definition it readily follows that an e-compactification of a space X is an H-closed extension. The following theorem shows that the converse need not be true.

THEOREM 2.2 **[He]**. *Let pX be an extension of X. Then the following statements are equivalent*:

(i) *pX is an e-compactification of X.*

(ii) *Every ultrafilter on X has an accumulation point in pX.*

(iii) *pX is H-closed and $X \cup \{q\}$ is regular, for all $q \in pX$.* $\square$

It follows that an e-compactifiable space is regular. The converse is not the case. From 2.2(iii) we can conclude that each noncompact $\mathfrak{R}$-closed space (i.e. a regular space which is closed in every regular space in which it is embedded, see **[BS]**) is an example of a regular non-e-compactifiable space. It is clear that every Tychonoff space is e-compactifiable, and in **[He]** the question appeared whether the converse

Received by the editors February 21, 1983.
1980 *Mathematics Subject Classification*. Primary 54C10, 54D20, 54G20; Secondary 54D25.
Key words and phrases. e-compactifiable spaces, perfect maps.

holds. In the next section we show that this is not the case. We were unable to characterize the class of e-compactifiable spaces in terms of some separation property.

The following properties of e-compactifiable spaces are known.

THEOREM 2.3 **[He]**. (i) *Let pX be an e-compactification of X. Then $\mathrm{cl}_{pX} Y$ is an e-compactification of Y, for each $Y \subset X$.*

(ii) *Let $p_i X_i$ be an e-compactification of X_i ($i \in I$). Then $\prod p_i X_i$ is an e-compactification of $\prod X_i$.* □

Recall that a subset $U \subset X$ is regular-closed if $\mathrm{cl}\,\mathrm{int}\, U = U$. The collection of regular-closed subsets of X is a closed base for some topology on X. X supplied with this topology is called the semiregularization of X, to be denoted by X_S. X is called semiregular if X is homeomorphic to X_S.

In **[St]** R. M. Stephenson observed that Theorem 2.3 implies that each e-compactifiable space X has a largest e-compactification eX, i.e. if αX is an e-compactification of X then the map id: $X \to \alpha X$ has a continuous extension over eX.

THEOREM 2.4. (i) **[St]** *Let X be an e-compactifiable space. Then X is an open subspace of eX and $eX - X$ is a closed discrete subspace of eX.*

(ii) *Let f: $X \to Y$ be a continuous map and assume that both X and Y are e-compactifiable. Then there is a continuous extension ef: $eX \to eY$ of f.*

PROOF. (ii) According to 2.3(ii) we have that $eX \times eY$ is an e-compactification of $X \times Y$. Define $\tilde{X} = \{(x, f(x)): x \in X\} \subset X \times Y$. $\tilde{X}$ is a closed subset of $X \times Y$ and $\prod_X \restriction \tilde{X}$: $\tilde{X} \to X$ is a homeomorphism. Since $\mathrm{cl}_{eX \times eY} \tilde{X}$ is an e-compactification of $\tilde{X}$, the map $(\prod_X \restriction \tilde{X})^{-1}$: $X \to \tilde{X}$ has an extension $e(\prod_X \restriction \tilde{X})^{-1}$: $eX \to \mathrm{cl}_{eX \times eY} \tilde{X}$. Define $ef = \prod_{eY} \circ e(\prod_X \restriction \tilde{X})^{-1}$. □

As a method to answer the question of S. Hechler, R. M. Stephenson asked the following question.

"Let X be an e-compactifiable space. Is the space $(eX)_S$ always compact?"

Our example of a non-Tychonoff e-compactifiable space provides a negative answer to this question. A partial positive answer to Stephenson's question is the following

THEOREM 2.5 **[St]**. *Let X be a regular space. If disjoint regular closed sets are contained in disjoint open subsets (in particular, if X is normal), then X is Tychonoff (hence e-compactifiable) and $(eX)_S$ is compact.* □

Our second example shows that the answer is negative if X is only assumed to be Tychonoff. The following simple lemma is one of the keys to the construction.

LEMMA 2.6. *Let X be a Tychonoff space. Then $(eX)_S$ is compact iff the map e(id): $eX \to \beta X$ is injective.*

PROOF. Observe that X is a subspace of $(eX)_S$ and that the map e(id): $(eX)_S \to \beta X$ is also continuous. Then we have " $\to$ ", since $(eX)_S$ is a compactification of X and " $\leftarrow$ " holds because $(eX)_S$ is minimal Hausdorff and the topology of βX is weaker than that of $(eX)_S$. □

3. The results. The following theorem is the key to our construction of a non-Tychonoff e-compactifiable space.

THEOREM 3.1. *Perfect preimages of e-compactifiable spaces are e-compactifiable.*

PROOF. Let X be an e-compactifiable space and let $f\colon Y \to X$ be a perfect map. We construct an e-compactification αY of Y in the following way. The underlying set of αY is $Y \oplus (eX - X)$ and a topology is defined by

(i) Y is open in αY;

(ii) For $p \in \alpha Y - Y = eX - X$ the collection $\mathcal{U}_p = \{\{p\} \cup f^{-1}(X \cap U)\colon U$ open in eX & $p \in U\}$ is taken as a local base in $p \in \alpha Y$.

One readily sees that αY is a Hausdorff extension of Y. To see that αY is an e-compactification of Y, consider an ultrafilter $\mathcal{F}$ on Y. Then $f(\mathcal{F}) = \{f(F)\colon F \in \mathcal{F}\}$ is an ultrafilter on X; hence $f(\mathcal{F})$ has an accumulation point q in eX. If $q \in X$ then, since f is perfect, $\mathcal{F}$ has an accumulation point in $f^{-1}(q)$. If $q \in eX - X$, then $f(F) \cap U_q \neq \varnothing$ for each open neighborhood U_q of q in eX and $F \in \mathcal{F}$. Since $f(F) \subset X$ it follows that $F \cap f^{-1}(U_q \cap X) \neq \varnothing$, i.e. q—considered as an element of αY—is an accumulation point of $\mathcal{F}$. $\square$

In **[Ch]** J. Chaber constructed examples of non-Tychonoff perfect preimages of Tychonoff spaces, and so these examples establish the existence of non-Tychonoff e-compactifiable spaces. From 2.3(i) it follows that subspaces of perfect preimages of Tychonoff spaces are e-compactifiable. We were not able to construct e-compactifiable spaces outside this particular class. Observe that a space X in this class (with $|X| > 1$) admits nonconstant real-valued continuous functions.

Question 3.2. Do there exist e-compactifiable spaces on which every real-valued continuous function is constant?

Let us now answer the question of R. M. Stephenson, whether there exist Tychonoff spaces X for which $(eX)_S$ is not compact. Our strategy is as follows. We construct a Tychonoff space X, a point $p \in \beta X - X$ and an extension αX of X such that $|\alpha X - X| > 1$ and such that the map $f\colon \alpha X \to X \cup \{p\}$ $(\subset \beta X)$ defined by $f(x) = x$ $(x \in X)$ and $f(\alpha X - X) = p$ is perfect. It then follows that αX is e-compactifiable, and since $e\alpha X$ can be considered as an e-compactification of X, we can conclude from the diagram below that the map $e(\mathrm{id})\colon eX \to \beta X$ is not injective. (e_1 is the extension of id: $X \to \alpha X \subset e\alpha X$ to eX (see 2.4(iii)).) (e_2 is the extension of id: $X \cup \{p\} \to \beta(X \cup \{p\})$ to $e(X \cup \{p\}$.) Indeed, the diagram shows that $e(\mathrm{id}) = e_2 \circ ef \circ e_1$; hence $e(\mathrm{id})^{-1}(p) > 1$.

$$\begin{tikzcd}
 & eX \arrow[drr, "e_1"] & & \\
 & \alpha X \arrow[d, "f"] & & e\alpha X \arrow[ll] \arrow[d, "ef"] \\
X \arrow[uur] \arrow[ur] \arrow[r] \arrow[dr] & X \cup \{p\} \arrow[d] & & e(X \cup \{p\}) \arrow[ll] \arrow[dll, "e_2"] \\
 & \beta X = \beta(X \cup \{p\}) & &
\end{tikzcd}$$

The example we present is almost identical to the one constructed by J. Chaber. The only difference lies in the fact that we want the point p to lie in the Čech-Stone remainder of X. For the reader's convenience we give the construction in detail.

EXAMPLE 3.3. Put $T = (\omega_1 + 1) \times (\omega_1 + 1) - \{(\omega_1, \omega_1)\}$. The set of pairs of the form $(\alpha, \omega_1) \in T$ will be called the left edge of T. The set of pairs of the form $(\omega_1, \alpha) \in T$ will be called the right edge of T. Define the space T'', for $n \in \mathbf{N}$, as the space obtained by identification in the sum $\bigoplus_{i=1}^{n} T(i)$ where $T(i) = T \times \{i\}$, of the right edge of $T(i)$ with the left edge of $T(i+1)$. Let $\varphi_n\colon \bigoplus_{i=1}^{n} T(i) \to T''$ be the corresponding identification-map. For each $0 \leqslant k \leqslant n$ we define an open subset $U_k^n \subset T''$, by

$$U_k^n = \begin{cases} \operatorname{int} \varphi_n(T(1)) & (k = 0), \\ \operatorname{int} \varphi_n(T(k) \cup T(k+1)) & (k = 1, \ldots, n-1), \\ \operatorname{int} \varphi_n(T(n)) & (k = n). \end{cases}$$

Finally we define $X = \bigoplus_{n=1}^{\infty} T''$.

It is well known that $|\beta T'' - T''| = 1$, for each $n \in \mathbf{N}$. For $\alpha < \omega_1$ put $Z_\alpha = [\alpha, \omega_1] \times [\alpha, \omega_1] - \{(\omega_1, \omega_1)\}$. Then $\{Z_\alpha\colon \alpha < \omega_1\}$ is a base for the unique nonfixed z-ultrafilter on T. If we define, for $n \in \mathbf{N}$, $Z_\alpha^n = \varphi_n(\bigoplus_{i=1}^{n}(Z_\alpha \times (i)))$ then $\{Z_\alpha^n\colon \alpha < \omega_1\}$ is a base for the unique nonfixed z-ultrafilter $\mathbf{Z}''$ on T''.

Next we define a point $p \in \beta X - X$. Let $\mathcal{G}$ be a nonfixed ultrafilter on $\mathbf{N}$. For $G \in \mathcal{G}$ and $\alpha < \omega_1$ put $Z(G, \alpha) = \bigcup\{Z_\alpha^n\colon n \in G\}$. It is easy to verify that the collection $\{Z(G, \alpha)\colon G \in \mathcal{G}, \alpha < \omega_1\}$ is a base for a nonfixed z-ultrafilter $\mathcal{F}$ on X. Let $p \in \beta X - X$ be the point in βX corresponding to $\mathcal{F}$, i.e. $\{p\} = \bigcap\{\mathrm{cl}_{\beta X} F\colon F \in \mathcal{F}\}$. In the space $X \cup \{p\}$ we have the following: If U is open in X then $U \cup \{p\}$ is a neighborhood of p in $X \cup \{p\}$ iff $\exists G \in \mathcal{G} \exists \alpha < \omega_1$ such that $Z(G, \alpha) \subset U$. $(*)$

(This is not completely trivial, since X is not normal. However, it follows easily by considering the space $\tilde{X} = \bigoplus_{n=1}^{\infty} \mathrm{cl}_{\beta X} T'' \subset \beta X$, which is σ-compact (hence normal). We omit the details.)

Let us now introduce a topology on the set $X \cup [0, 1]$ ($[0, 1]$ is the unit interval) in the following way. For $t \in [0, 1]$ let $\{V_l(t)\}_{l=1}^{\infty}$ be a countable local base at t. For $t \in [0, 1]$, $l \in \mathbf{N}$, $G \in \mathcal{G}$, $\alpha < \omega_1$ define

$$U(t, l, G, \alpha) = \bigcup_{n \in G} \bigcup_{s \in V_l(t)} \left(U_{[n.s]}^n \cap Z_\alpha^n\right) \cup V_l(t).$$

(Here $[n.s]$ denotes the greatest integer not greater than $n.s$.) And next we put:

X is open in $X \cup [0, 1]$.

For $t \in [0, 1]$ the collection $\{U(t, l, G, \alpha)\colon l \in \mathbf{N}, G \in \mathcal{G}, \alpha < \omega_1\}$ is defined to be a local base of t in $X \cup [0, 1]$.

Observe that $[0, 1]$ is embedded in $X \cup [0, 1]$. It is easy to check that $X \cup [0, 1]$ is a Hausdorff space. In fact our topology has more open sets than Chaber's.

Claim. Let U be a subset of $X \cup [0, 1]$ which contains $[0, 1]$. Then U is neighborhood of $[0, 1]$ in $X \cup [0, 1]$ iff $\exists G \in \mathcal{G} \exists \alpha < \omega_1$ such that $Z(G, \alpha) \subset U$.

PROOF. Assume $Z(G, \alpha) \subset U$. Then, for $t \in [0,1]$, $t \in U(t, l, G, \alpha) \subset Z(G, \alpha)$. Hence $[0,1] \subset \operatorname{int} U$. On the other hand, assume $[0,1] \subset \operatorname{int} U$. Then, $\forall t \in [0,1]$ $\exists l(t) \in \mathbf{N}\, \exists G(t) \in \mathcal{G}\, \exists \alpha(t) < \omega_1$ such that

$$t \in U(t, l(t), G(t), \alpha(t)) \subset U.$$

Since $[0,1]$ is compact, $[0,1]$ can be covered by finitely many of these sets. Say $[0,1] \subset \bigcup_{i=1}^{k} U(t_i, l(t_i), G(t_i), \alpha(t_i))$ $(\subset U)$. Put $G = \bigcap_{i=1}^{k} G(t_i)$ $(\in \mathcal{G})$ and $\alpha = \sup\{\alpha(t_i)\colon i \leqslant k\}$ $(< \omega_1)$. We claim that $Z(G, \alpha) \subset \bigcup_{i=1}^{k} U(t_i, l(t_i), G(t_i), \alpha(t_i))$ $(\subset U)$. Choose $p \in Z(G, \alpha)$, say $p \in Z_\alpha^n$ for some $n \in G$. Since $T^n = \bigcup_{k=0}^{n} U_k^n$, there exists $k \leqslant n$ such that $p \in U_k^n$. Choose $s \in [0,1]$ such that $[n.s] = k$. If $s \in U(t_i, l(t_i), G(t_i), \alpha(t_i))$ then, since $G \subset G(t_i)$ and $Z_\alpha \subset Z_{\alpha(t_i)}$, we conclude that $p \in Z(\alpha(t_i)) \cap U_{[n.s]}^n$ for "some" $n \in G(t_i)$, i.e. $p \in U(t_i, l(t_i), G(t_i), \alpha(t_i))$. The claim follows.

From the claim and from $(*)$ we conclude that the space obtained from $X \cup [0,1]$ by identifying $[0,1]$ to a point is homeomorphic to $X \cup \{p\}$. Obviously the map f: $X \cup [0,1] \to X \cup \{p\}$ defined by $f(x) = x$ $(x \in X)$ and $f[0,1] = p$ is a perfect map. Hence, all the required properties are satisfied. □

REMARK 3.4. It is well known that each space T^n, as defined in 3.3, has a unique (nontrivial) regular extension, namely βT^n. It follows that $\mathrm{cl}_{eX} T^n \simeq \beta T^n$, for all $n \in \mathbf{N}$. Consider the space $\tilde{X} = \bigoplus_{n=1}^{\infty} \beta T^n$. Then $X \subset \tilde{X} \subset eX$. $\tilde{X}$ is a σ-compact, hence normal, and according to 2.5 this implies that $(e\tilde{X})_S = \beta\tilde{X} = \beta X$. Since $(eX)_S \neq \beta X$, we conclude that the map id: $\tilde{X} \to eX$ cannot be extended continuously to $e\tilde{X}$. At first glance this may seem a contradiction, but it is not. One cannot use 2.4(ii) to ensure that such an extension should exist since eX is not e-compactifiable (eX is not even semiregular), nor the fact that $e\tilde{X}$ is the largest e-compactification, since eX is not an e-compactification of $\tilde{X}$. (A nonfixed ultrafilter on $\tilde{X} - X$ does not have an accumulation point in eX.)

REFERENCES

[**BS**] M. P. Berri and R. H. Sorgenfrey, *Minimal regular spaces*, Proc. Amer. Math. Soc. **14** (1963), 454–458.

[**Ch**] J. Chaber, *Remarks on open-closed mappings*, Fund. Math. **73** (1971), 197–208.

[**He**] S. H. Hechler, *On a notion of weak compactness in non-regular spaces*, Studies in Topology (N. M. Stavrakas and K. R. Allen, eds.), Academic Press, New York, 1975, pp. 215–237.

[**St**] R. M. Stephenson, *Not every minimal Hausdorff space is e-compact*, Proc. Amer. Math. Soc. **52** (1975), 381–388.

SUBFACULTEIT WISKUNDE EN INFORMATICA, VRIJE UNIVERSITEIT, DE BOELELAAN 1081, 1081 HV AMSTERDAM, THE NETHERLANDS (Current address of K. P. Hart)

Current address (J. Vermeer): Department of Mathematics, University of Kansas, Lawrence, Kansas 66045