NOTE ON ROTATION SET

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Abstract. Let $f$ be an endomorphism of the circle of degree 1 and $\tilde{f}$ be a lifting of $f$. We characterize the rotation set $\rho(\tilde{f})$ by the set of probability measures on the circle, and prove that if $\rho_+(\tilde{f})$ ($\rho_-(\tilde{f})$), the upper (lower) endpoint of $\rho(\tilde{f})$, is irrational, then $\rho_+ (\rho_\theta (\tilde{f})) > \rho_+ (\tilde{f})$ ($\rho_- (\rho_\theta (\tilde{f})) > \rho_- (\tilde{f})$) for any $\theta > 0$, where $\rho_\theta (x) = x + \theta$. As a corollary, if $f$ is structurally stable, then both $\rho_+ (\tilde{f})$ and $\rho_- (\tilde{f})$ are rational.

Newhouse, Palis and Takens [3] have generalized a rotation number for a homeomorphism of the circle to a continuous map of degree 1 and defined a rotation set. Let $R$ denote the real numbers, $Z$ the integers, $N$ the positive integers and $\mathbb{S} = R/Z$ the circle. Let $\pi: R \to \mathbb{S}$ denote the canonical projection. Let $f: \mathbb{S} \to \mathbb{S}$ be a given continuous map of degree 1. Choose a lifting $\tilde{f}: R \to R$, that is a map such that $\pi \tilde{f} = f \pi$. Liftings exist and are unique up to the addition of an integer. Each lifting satisfies $\tilde{f}(x + 1) = \tilde{f}(x) + 1$.

Definition. Given $x \in R$, define the rotation number

$$\rho(\tilde{f}, x) = \limsup_{n \to \infty} \frac{1}{n} (\tilde{f}^n(x) - x).$$

Define the rotation set to be $\rho(\tilde{f}) = \{\rho(\tilde{f}, x) | x \in R\}$.

Notice that if a different lifting is used, this merely has the effect of translating the rotation set by an integer.

We recall the following properties of $\rho(\tilde{f})$.

1. If $f = hgh^{-1}$ for an orientation preserving homeomorphism $h$ of $\mathbb{S}$, then $\rho(\tilde{f}) = \rho(\tilde{g})$ for suitable liftings $\tilde{f}$ and $\tilde{g}$.

2. $\rho(\tilde{f})$ is either one point or a closed interval.

3. For any $\alpha \in \rho(\tilde{f})$, there exists $x \in R$ such that $\lim_{n \to \infty} (f^n(x) - x)/n = \alpha$.

(1) is trivial from the definition. See [2] and [3] for (2), (3) and other elementary properties of $\rho(\tilde{f})$. By (2) we may denote the upper and lower endpoints of $\rho(\tilde{f})$ by $\rho_+ (\tilde{f})$ and $\rho_- (\tilde{f})$ respectively.

The aim of this paper is to prove the following

Theorem 1. Let $M$ be the set of probability measures on $\mathbb{S}$ invariant with respect to $f$. Let $\varphi = \tilde{f} - \text{Id}: \mathbb{S} \to R$ where $\text{Id}: R \to R$ denotes identity. Then $\rho(\tilde{f}) = \{\mu(\varphi) | \mu \in M\}$.

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Notes. (i) The set of probability measures on $S$ is regarded as the set of positive linear functionals $\mu$ on $C(S)$ such that $\mu(1) = 1$.

(ii) $S$ is considered to be $[0, 1]$ with 0 and 1 identified, and $\phi(x) = \hat{f}(x) - x$ for $x \in [0, 1]$.

**Theorem 2.** If $\rho_\theta (\hat{f})$ is irrational and $\theta > 0$, then $\rho_\theta (R_\theta \hat{f}) > \rho_\theta (\hat{f})$ (and $R_\theta$ is defined as $R_\theta(x) = x + \theta$).

These two theorems are generalizations of properties given in Herman [1] for the case $f$ is a homeomorphism of $S$.

Considering (1) and Theorem 2, we obtain the following

**Corollary.** If $f$ is structurally stable, then both $\rho_\theta (\hat{f})$ and $\rho_{-\theta} (\hat{f})$ are rational.

**Proof of Theorem 1.** By (3) above, for any $\alpha \in \rho(f)$, there exists $x \in S$ satisfying

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \phi(f^i(x)) = \lim_{n \to \infty} \frac{1}{n} (\hat{f}^n(x) - x) = \alpha
\]

where $\phi = \hat{f} - \text{Id}$.

Define $\mu_n : C(S) \to R$ by $\mu_n(g) = n^{-1} \sum_{i=0}^{n-1} g(f^i(x))$ for $g \in C(S)$ using $x$ above. Then $\mu_n$ is a probability measure on $S$. Since the set of probability measures is weak* compact, there is a subsequence $\{\mu_{n_k}\}$ of $\{\mu_n\}$ which converges weakly to a probability measure, say $\mu$. That is, we have $\mu_{n_k}(g) \to \mu(g)$ for any $g \in C(S)$. Taking $\phi = f - \text{Id}$ for $g$, we have $\mu_{n_k}(\phi) \to \mu(\phi)$, while by definition,

\[
\lim_{n \to \infty} \mu_{n_k}(\phi) = \lim_{n \to \infty} \frac{1}{k_n} \sum_{i=0}^{k_n-1} \phi(f^i(x)) = \alpha.
\]

Therefore, we have $\mu(\phi) = \alpha$. Since $\lim_{n \to \infty} \mu_{n_k}(g) = \mu(g)$, $\mu$ is invariant with respect to $f$.

Consequently $\rho(\hat{f}) \subset \{\mu(\phi) | \mu \in M\}$. On the other hand, for any rational $p/q < \rho_\theta (\hat{f})$, we have $\hat{f}^q(x) - x - p > 0$ for any $x \in S$. Thus $\mu(\hat{f}^q - \text{Id} - q \cdot \phi) > 0$ for any $\mu \in M$, while

\[
\mu(\hat{f}^q - \text{Id} - q\phi(\phi)) = \mu \left( \sum_{i=0}^{q-1} (\hat{f}^q - \text{Id}) \circ f^i \right) - q\mu(\phi) = 0.
\]

Therefore $\mu(\phi) > p/q$.

By the same reasoning, we have $\mu(\phi) < p/q$ for any rational $p/q > \rho_{-\theta} (\hat{f})$. Since $\rho(\hat{f})$ is a closed set and $\rho(\hat{f}) \subset \{\mu(\phi) | \mu \in M\}$, we have $\rho(\hat{f}) = \{\mu(\phi) | \mu \in M\}$.

We need two lemmas to prove Theorem 2.

**Lemma 1.** For any irrational number $\alpha$ there exists a monotone decreasing sequence $\{p_n/q_n\}$ (and a monotone increasing sequence $\{p'_n/q'_n\}$) of rationals converging to $\alpha$ and satisfying $p_n/q_n - \alpha < 1/q_n^2$ (and $p'_n/q'_n < 1/q_n^2$).

As this is a well-known fact in arithmetic, we do not give the proof.
Lemma 2. Let $\theta > 0$. For any $k \in \mathbb{N}$ and any $x \in \mathbb{R}$, there exists $y \in \mathbb{R}$ such that $y \leq x$ and $(R_{\theta} \hat{f})^k(y) \geq \hat{f}^k(x) + \theta$.

Proof. We prove this by induction. For $k = 1$ it is trivial. Assume the lemma is true for $k$, then we have $y \leq x$ and $(R_{\theta} \hat{f})^k(y) \geq \hat{f}^k(x) + \theta$. Since $(R_{\theta} \hat{f})^k(s + n) = (R_{\theta} \hat{f})^k(s) + n$ for $n \in \mathbb{Z}$, we have $z \leq y$ such that $(R_{\theta} \hat{f})^k(z) = \hat{f}^k(x)$, then

$$(R_{\theta} \hat{f})^{k+1}(z) = \hat{f}((R_{\theta} \hat{f})^k(z)) + \theta = \hat{f}(\hat{f}^k(x)) + \theta = \hat{f}^{k+1}(x) + \theta$$

and $z \leq x$, completing the induction.

Proof of Theorem 2. We prove the theorem for $\rho_+(\hat{f})$ because the $\rho_-(\hat{f})$ case is similar. By Lemma 1, we may choose a sequence $\{p_n/q_n\}$ of rationals such that $\{p_n/q_n\} \downarrow \alpha$ and $p_n/q_n - 1/q_n^2 < \alpha$. Since $\rho_+(\hat{f}) = \alpha$, we have $\hat{f}^{q_n}(x) - x < p_n$ for any $x \in \mathbb{R}$. Suppose that there exists $\theta > 0$ such that $\hat{f}^{q_n}(x) - x < p_n - \theta$ for any $n \in \mathbb{N}$ and any $x \in \mathbb{R}$. Take $q_n$ large enough to satisfy $q_n \theta > 1$, then we have

$$\hat{f}^{q_n}(x) - x = \sum_{i=0}^{q_n-1} (\hat{f}^{q_n} \circ \hat{f}^{i)(x)} - \hat{f}^{i+1}(x)) < q_n(p_n - \theta) < q_n p_n - 1.$$

Thus, we have $\rho_+(\hat{f}) < (q_n p_n - 1)/q_n^2 < \alpha$, contradicting $\rho_+(\hat{f}) = \alpha$. Therefore, for any $\theta > 0$, there exist $n \in \mathbb{N}$ and $x \in \mathbb{R}$ such that $\hat{f}^{q_n}(x) - x \geq p_n - \theta$. On the other hand, by Lemma 2, there exists $y \leq x$ satisfying $(R_{\theta} \hat{f})^{q_n}(y) \geq \hat{f}^{q_n}(x) + \theta$. Thus, we have $(R_{\theta} \hat{f})^{q_n}(y) - y > \hat{f}^{q_n}(x) - x + \theta \geq p_n$. Therefore, $\rho_+(R_{\theta} \hat{f}) \geq p_n/q_n > \alpha$, completing the proof.

References


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