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A FORMULA FOR DISTRIBUTION TRACE CHARACTERS ON NILPOTENT LIE GROUPS

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ABSTRACT. The distribution trace character θ_π of an irreducible representation π on a simply connected nilpotent Lie group N is described as a superposition of unitary characters on certain subgroups, in terms of the canonical objects introduced by R. Penney [8]. For $l \in \mathfrak{n}^*$, define $\mathfrak{h}_1(l)$ = smallest ideal containing the radical $\mathfrak{r}(l)$, and $\mathfrak{h}_{k+1}(l) = \mathfrak{h}_1(l \mid \mathfrak{h}_k(l))$. These subalgebras terminate in a subordinate subalgebra $\mathfrak{h}_\infty(l)$ after finitely many steps. If $H_\infty = \exp(\mathfrak{h}_\infty)$, $\chi_\infty = (e^{2\pi i l}) \circ \log | H_\infty$, and $(\chi_\infty, H_\infty) \cdot n = (\chi_\infty \cdot n, H_\infty \cdot n)$, where $H_\infty \cdot n = n^{-1} H_\infty n$, $\chi_\infty \cdot n(h') = \chi_\infty(nh'n^{-1})$ on $H_\infty \cdot n$, then $\langle \theta_\pi, \phi \rangle = \int_{K_\infty \backslash N} \langle (\chi_\infty, H_\infty) \cdot n, \phi \rangle \, d\dot{n}$, where the pair $(\chi_\infty, H_\infty) \cdot n$ is regarded as the tempered distribution $\langle (\chi_\infty, H_\infty) \cdot n, \phi \rangle = \int_{H_\infty} \chi_\infty(h) \phi(n^{-1} h n) \, dh$, and where $\mathfrak{k}_\infty = \{X \in \mathfrak{n} : l[h_\infty, X] = 0\}$ gives the stabilizer of the pair (χ_∞, H_∞). The integral over $K_\infty \backslash N$ is absolutely convergent for any Schwartz function ϕ on N.

If N is a simply connected nilpotent Lie group and $\pi \in \hat{N}$ an irreducible representation, then π_ϕ is trace class for any $\phi \in C_c^\infty(N)$ and the trace character $\langle \theta_\pi, \phi \rangle = \operatorname{Tr} \pi_\phi$ is a tempered distribution on N. Its Fourier transform has been described in terms of coadjoint orbits in $\mathfrak{n}^*$ by Kirillov [6]. Various attempts have been made to describe the θ_π directly as distributions on N, but this seems quite difficult. Dixmier [3] computed a number of examples, and some asymptotic properties were discussed in [1] for special N.

This note shows how θ_π can be represented as a superposition of ordinary characters on certain canonical subgroups on N related to the canonical objects introduced by Penney in [8], and exploited in [2]. The formula we get is clearly suggested by [2], but a very direct calculation can be made for the θ_π. This formula can be used to estimate the order of θ_π as a Schwartz distribution, and may prove to have uses in the harmonic analysis of smooth functions on N.

Let $\mathcal{S}(N)$ be the Schwartz functions on N (i.e. $f \circ \exp$ is Schwartz on the Lie algebra $\mathfrak{n}$). If $\pi \in \hat{N}$, let l be an element in the coadjoint orbit $\mathcal{O} = \mathcal{O}_\pi \subseteq \mathfrak{n}^*$, $\mathfrak{m}$ a maximal subordinate subalgebra for l, and $\chi = (e^{2\pi i l}) \circ \log | M$ the resulting character on $M = \exp(\mathfrak{m})$. Let $\{X_1, \ldots, X_m, \ldots, X_n\}$ be a Jordan-Hölder basis for $\mathfrak{n}$ that passes through $\mathfrak{m}$; $\mathfrak{n}_j = \mathbf{R} - \operatorname{span}\{X_1, \ldots, X_j\}$ a subalgebra with $\mathfrak{n}_j \lhd \mathfrak{n}_{j+1}$ for all j, and $\mathfrak{n}_m = \mathfrak{m}$. The map $\gamma: \mathbf{R}^k \to N$,

$$\gamma(t) = \exp(t_1 X_{m+1}) \cdots \exp(t_k X_{m+k}), \qquad k = n - m,$$

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cross sections the $M \backslash N$ cosets, carries Lebesgue measure dt to right invariant measure $d\dot{n}$ on $M \backslash N$, and allows us to realize the Hilbert space of the induced representation $\pi = \operatorname{Ind}(M \uparrow N, \chi)$ isometrically as $L^2(\mathbf{R}^k)$, as follows: the function $\tilde{f}$ on G corresponds to $f \in L^2(\mathbf{R}^k)$ via restriction to the cross-section $\gamma(\mathbf{R}^k)$ and pullback to $\mathbf{R}^k$,

$$\tilde{f}(n) = \tilde{f}(m\gamma(t)) = \chi(m)f(t).$$

For $\phi \in \mathcal{S}(N)$ we may explicitly compute the kernel $K_\phi(s, t)$ of the operator π_ϕ:

$$\begin{aligned}
\pi_\phi f(t) = \pi_\phi \tilde{f}(\gamma(t)) &= \int_N \phi(n)\pi_n \tilde{f}(\gamma(t))\,dn = \int_N \phi\big(\gamma(t)^{-1}n\big)\tilde{f}(n)\,dn \\
&= \int_{\mathbf{R}^k}\int_M \phi\big(\gamma(t)^{-1}m\gamma(s)\big)\tilde{f}(m\gamma(s))\,dm\,ds \\
&= \int_{\mathbf{R}^k}\left[\int_M \chi(m)\phi\big(\gamma(t)^{-1}m\gamma(s)\big)\,dm\right] f(s)\,ds \\
&= \int_{\mathbf{R}^k} K_\phi(t, s)f(s)\,ds.
\end{aligned}$$

Since π_ϕ is trace class [**9**, p. 108 ff.] and K_ϕ is continuous, the trace is just the integral of the diagonal values

$$(1) \qquad \langle \theta_\pi, \phi \rangle = \operatorname{Tr} \pi_\phi = \int_{\mathbf{R}^k} K_\phi(t, t)\,dt$$

and the integral is absolutely convergent (see [**4**, III.10] for example).

Let $(\chi, M) \cdot n = (\chi \cdot n, M \cdot n)$, where $M \cdot n = n^{-1}Mn$ and $\chi \cdot n(m') = \chi(nm'n^{-1})$ for $m' \in M \cdot n$, to obtain a right action of N on pairs (χ, M) where χ is a character on M. Fixing a Haar measure dm on M, we may regard each pair in an N orbit as a tempered distribution

$$\langle (\chi, M) \cdot n, \phi \rangle = \int_M \chi(m)\phi(n^{-1}mn)\,dm, \quad \text{all } \phi \in \mathcal{S}(N).$$

From (1) and the fact that $(\chi, M) = (\chi, M) \cdot n \Leftrightarrow n \in M$, we see that θ_π is a superposition of all pairs in the orbit $(\chi, M) \cdot N$:

$$\begin{aligned}
(2) \qquad \langle \theta_\pi, \phi \rangle = \int_{\mathbf{R}^k} K_\phi(t, t)\,dt &= \int_{\mathbf{R}^k}\left[\int_M \chi(m)\phi\big(\gamma(t)^{-1}m\gamma(t)\big)\,dm\right] dt \\
&= \int_{M \backslash N} \langle (\chi, M) \cdot n, \phi \rangle\,d\dot{x},
\end{aligned}$$

the formula being absolutely convergent for all $\phi \in \mathcal{S}(N)$. Formula (2) is not canonical since it involves a choice of l and a polarization $\mathfrak{m}$. It must involve a great deal of hidden cancellation, as the canonical version below indicates.

For $l \in \mathfrak{n}^*$ we define, following Penney [**8**], the canonical objects: $\mathfrak{h}_1(l)$ = smallest ideal containing the radical $\mathfrak{r}(l)$, and $\mathfrak{h}_{k+1}(l) = \mathfrak{h}_1(l \,|\, \mathfrak{h}_k(l))$. These subalgebras stabilize eventually, $\mathfrak{h}_1 \supseteq \mathfrak{h}_2 \supseteq \cdots \supseteq \mathfrak{h}_k = \mathfrak{h}_{k+1} = \cdots \mathfrak{h}_\infty$; all contain the radical

$\mathfrak{r}(l)$ and are transformed covariantly:

$$\mathfrak{h}_k(l \cdot n) = \mathfrak{h}_k(l) \cdot n \quad \text{where } l \cdot n = \mathrm{Ad}^*(n^{-1})l \text{ and } X \cdot n = \mathrm{Ad}(n^{-1})X,$$

and $\mathfrak{h}_\infty$ is subordinate to l. Let $\mathfrak{k}_\infty(l) = \{X \in \mathfrak{n}: l[\mathfrak{h}_\infty, X] = 0\}$; this turns out to be a subalgebra, and if we set $l_\infty = l | \mathfrak{k}_\infty$, then $\mathfrak{r}(l_\infty) = \mathfrak{h}_\infty(l) \lhd \mathfrak{k}_\infty(l)$ so that $l_\infty \in \mathfrak{k}_\infty^*$ has flat orbit (see [7]). Let $\chi_\infty = (e^{2\pi i l}) \circ \log | H_\infty$; by fixing a Haar measure dh on H_∞ we may regard (χ_∞, H_∞) and all its conjugates $(\chi_\infty, H_\infty) \cdot n$ as tempered distributions. The significance of the following formula is that $\mathfrak{h}_\infty$ can be much smaller than any polarization $\mathfrak{m}$ for l.

THEOREM 1. *Let $l \in \mathfrak{n}^*$ and define H_∞, K_∞, χ_∞ as above. Fixing dh on H_∞ and $d\dot{n}$ on $K_\infty \backslash N$, and regarding each $(\chi_\infty, H_\infty) \cdot n$ as a tempered distribution*

$$\langle (\chi_\infty, H_\infty) \cdot n, \phi \rangle = \int_{H_\infty} \chi(h) \phi(n^{-1}hn)\, dh, \qquad \phi \in \mathbb{S}(N),$$

we get

$$\langle \theta_\pi, \phi \rangle = \int_{K_\infty \backslash N} \langle (\chi_\infty, H_\infty) \cdot n, \phi \rangle \, d\dot{n}, \quad \text{all } \phi \in \mathbb{S}(N), \tag{3}$$

where $d\dot{n}$ is suitably normalized. The integral is absolutely convergent.

PROOF. Recall that $K_\infty = \{x \in N: (\chi_\infty, H_\infty) \cdot n = (\chi_\infty, H_\infty)\}$ (see [**2**]). Let M be any maximal subordinate subgroup for $l_\infty \in \mathfrak{k}_\infty^*$; it must also be maximal subordinate for $l \in \mathfrak{n}^*$, by dimension counting. Let $\chi = (e^{2\pi i l}) \circ \log | M$ and $\pi = \mathrm{Ind}(M \uparrow N, \chi)$. We start with a remark on the special case: π square integrable modulo $Z(N)$. This is precisely the case in which $H_\infty(l) = R(l) = Z(N)$. As is well known [7], we have flat orbit $\mathcal{O}(l) = l + \mathfrak{z}^\perp$. From a simple version of Kirillov's orbital integral formula and Poisson summation, we get

$$\begin{aligned} \theta_\pi &= \text{Euclidean Fourier transform of Lebesgue measure on } l + \mathfrak{z}^\perp \\ &= \text{scalar multiple of } \chi_\infty \text{ weighting Haar measure on } Z(N), \end{aligned}$$

or $\langle \theta_\pi, \varphi \rangle = \langle (\chi_\infty, H_\infty), \varphi \rangle$ where dh on H_∞ is suitably chosen. Since $\mathfrak{k}_\infty = \mathfrak{n}$ here, we have proved (3) in this special case.

We now use the noncanonical formula (2) to complete the proof of (3).

Take an arbitrary $l \in \mathfrak{n}^*$, consider a system $H_\infty \subseteq M \subseteq K_\infty$, and let

$$\pi_\infty = \mathrm{Ind}(M \uparrow K_\infty, \chi) \in \hat{K}_\infty.$$

Applying (2) to this situation, we get

$$\langle \theta_{\pi_\infty}, \phi \rangle = \int_{M \backslash K_\infty} \langle (\chi, M) \cdot k, \phi \rangle \, d\dot{k}, \quad \text{all } \phi \in \mathbb{S}(K_\infty) \tag{4}$$

(or $\phi \in \mathbb{S}(N)$, for that matter). We claim that

$$\langle \theta_{\pi_\infty}, \phi \rangle = \langle (\chi_\infty, H_\infty), \phi \rangle, \quad \text{all } \phi, \tag{5}$$

so the superposed conjugates of (χ, M) by elements in K_∞ collapse to the single character (χ_∞, H_∞) through cancellations. Once (5) is proved, we finish proving (3) as follows: by Fubini,

$$\begin{aligned}\langle \theta_\pi, \phi \rangle &= \int_{M\setminus N} \langle (\chi, M)\cdot n, \phi\rangle \, d\dot{n} \\ &= \int_{K_\infty\setminus N}\left[\int_{M\setminus K_\infty} \langle (\chi, M)\cdot k, \phi \circ \mathrm{Ad}(x)\rangle \, d\dot{k}\right] d\dot{x} \\ &= \int_{K_\infty\setminus N} \langle (\chi_\infty, H_\infty), \phi\circ \mathrm{Ad}(x)\rangle \, d\dot{x} = \int_{K_\infty\setminus N}\langle (\chi_\infty, H_\infty)\cdot x, \phi\rangle \, d\dot{x}\end{aligned}$$

with all integrals absolutely convergent.

To prove (5) it suffices to prove the following general statement about functionals $l \in \mathfrak{n}^*$ with flat orbit (i.e. $\mathfrak{r}(l) \lhd \mathfrak{n}$, so that $H_\infty = R(l)$, $K_\infty = N$ and $\pi_\infty = \pi = \mathrm{Ind}(M \uparrow N, \chi)$).

(6) If $l \in \mathfrak{n}^*$ has $\mathfrak{r}(l) \lhd \mathfrak{n}$, $\mathfrak{m}$ is any maximal subordinate subalgebra, $\chi = (e^{2\pi i l}) \circ \log | M$, and $\pi = \mathrm{Ind}(M \uparrow N, \chi)$, then $\theta_\pi = (\chi_\infty, H_\infty)$.

Here the orbit is flat, and $\mathcal{O}_\pi = l + \mathcal{H}_\infty^\perp$; the N-invariant measure on $\mathcal{O}_\pi$ is just Euclidean measure on $\mathcal{H}_\infty^\perp$ lifted to this coset. By Poisson summation and Kirillov's description of θ, we get (6). This completes the proof of Theorem 1. Q.E.D.

The canonical formula (3) leads to the following conclusion about flat parts in coadjoint orbits. If $\mathfrak{m}$ is maximal subordinate for $l \in \mathfrak{n}^*$, it is well known that $\mathcal{O}$ is a disjoint union of the flat fibers $l \cdot x + m^\perp \cdot x = l \cdot Mx$, $\dot{x} \in M\setminus N$. In general $\mathfrak{h}_\infty \subseteq \mathfrak{m}$, so $\mathfrak{h}_\infty^\perp$ may be larger than $\mathfrak{m}^\perp$. From (3) we see that

$$\mathcal{O} = \cup \left\{ (l + \mathfrak{h}_\infty^\perp) \cdot x \colon x \in K_\infty \setminus N \right\}.$$

This is a disjoint union: if $x, y \in N$ give nondisjoint fibers, then $(l + \mathfrak{h}_\infty^\perp) \cdot xy^{-1}$ meets $l + \mathfrak{h}_\infty^\perp$. But by dimension counting we can show that $l + \mathfrak{h}_\infty^\perp = l \cdot K_\infty$, so there exist $k_1, k_2 \in K_\infty$ such that $l \cdot k_1(xy^{-1})k_2 = l$. Thus $k_1(xy^{-1})k_2 \in R(l) \subseteq H_\infty \subseteq K_\infty$, and $xK_\infty = yK_\infty$ as required. Formula (3) also shows that $\mathrm{supp}(\theta_\pi) \subseteq$ the smallest connected (normal) subgroup containing $\cup_{n\in N} n^{-1}H_\infty n$. Since $\mathfrak{h}_1 =$ ideal generated by $\mathfrak{r}(l) \supseteq \mathfrak{h}_\infty \supseteq \mathfrak{r}(l)$, this amounts to saying that $\mathrm{supp}(\theta_\pi) \subseteq H_1$. By other means, Pukanszky **[10]** has recently shown that a normal Lie subgroup H contains $\mathrm{supp}(\theta_\pi) \Leftrightarrow H \supseteq H_1$, so further reduction of the support through cancellation cannot occur.

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