Lie Ideals and Jordan Derivations of Prime Rings

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Abstract. Herstein proved [1, Theorem 3.3] that any Jordan derivation of a prime ring of characteristic not 2 is a derivation of R. Our purpose is to extend this result on Lie ideals. We prove the following.

Theorem. Let R be any prime ring such that char R ≠ 2 and let U be a Lie ideal of R such that \( u^2 \in U \) for all \( u \in U \). If \( \cdot \), is an additive mapping of R into itself satisfying \( (u^2)' = u'u + uu' \) for all \( u \in U \), then \( (uv)' = u'v + uv' \) for all \( u, v \in U \).

Introduction. Herstein [1, Theorem 3.3] proved that if \( R \) is a prime ring of characteristic different from 2, then any Jordan derivation of \( R \), i.e., an additive mapping of \( R \) into itself such that \( (a^2)' = a'a + aa' \) for all \( a \in R \), is a derivation of \( R \), i.e., an additive mapping of \( R \) into itself such that \( (ab)' = a'b + ab' \) for all \( a, b \in R \). In this paper we generalize this result on Lie ideals.

Throughout the paper we assume \( R \) is a prime ring of characteristic not 2. The center of \( R \) is denoted by \( Z \). We always assume \( U \) is a Lie ideal of \( R \) with the condition that \( u^2 \in U \) for all \( u \in U \). We also assume, \( \cdot \), is an additive mapping of \( R \) into itself such that

\( (u^2)' = u'u + uu' \) for all \( u \in U \).

Note that \( (uv + vu) = (u + v)^2 - (u^2 + v^2) \). Hence \( (uv + vu) \) is in \( U \) and condition (i) is equivalent to

\( (uv + vu)' = u'v + uv' + v'u + vu' \) for all \( u, v \in U \).

For \( x, y \in R \), let

\[ [x, y] = xy - yx \quad \text{and} \quad x^y = (xy)' - x'y - xy'. \]

If \( A \) is a subset of \( R \), we define the centralizer of \( A \)

\[ C_R(A) = \{ x \in R | [x, a] = 0 \text{ for all } a \in A \}. \]

An additive subgroup \( U \) of \( R \) is said to be a Lie ideal of \( R \) if \( [u, r] \in U \) for all \( u \in U \) and \( r \in R \). For the remainder of the paper, the letters \( u, v, w, u_i, v_i, w_i \) will always denote arbitrary elements in \( U \).

If \( U \) is a commutative Lie ideal of \( R \), then by the proof of Lemma 1.3 [1], \( U \subset Z \). Then from (ii) we get \( 2(uv)' = 2(u'v + uv') \). Since \( \text{char } R \neq 2 \), we get the desired conclusion.

Thus we shall always assume \( U \) is a noncommutative Lie ideal of \( R \), i.e., \( U \not\subset Z \).
2. Basic lemmas.

**Lemma 1.** If \( U \not\subseteq Z \) is a Lie ideal of \( R \), then
\[
(uvu)' = u'vu + uv'u + uvu'
\]
for all \( u, v \in U \).

**Proof.** The proof is the same as that of Lemma 3.5 of [1], since \( uv + vu \in U \) for \( u, v \in U \).

By linearizing Lemma 1 on \( u \), we get

**Lemma 2.** If \( U \not\subseteq Z \) is a Lie ideal of \( R \), then
\[
(uvw + wvu)' = u'vw + uv'w + uvw' + w'vu + wvu' + wvu'
\]
for all \( u, v, w \in W \).

**Lemma 3.** If \( U \not\subseteq Z \) is a Lie ideal of \( R \), then \( u^c[u, v] = 0 \) for all \( u, v \in U \).

**Proof.** Since for any \( u, v \in U, uv + vu \in U \) and also \( uv - vu \in U \), as \( U \) is a Lie ideal, we have \( 2uv \in U \). Therefore, from (i), since \( \text{char } R \neq 2 \), we get
\[
((uv)^2)' = (uv)'(uv) + (uv)(uv)'.
\]
In Lemma 2 replace \( w \) by \( 2uv \) to get
\[
2(uv(uv) + (uv)vu)' = 2((u'v^c(uv) + uv'(uv) + uv(uv)')
\]
\[
+ (uv)vu + (uv)v'u + (uv)v'w + (uv)w'u + (uv)w'v + w'vu + wvu')
\]
\[
= 2( (u'v + uv')w + (uv)'v'v + 2uv((uv)' + v'u + u'v')).
\]

But
\[
2(uv(uv) + (uv)vu)' = 2((uv)^2 + uv^2u)'
\]
\[
= 2((uv)'uv + uv(uv)') + u'v^2u + u(v'v + v'v' + v'v') + uv^2u')
\]
\[
= 2(((uv)'uv + (u'v + uv')vu) + 2uv((uv)' + v'u + u'v')).
\]
by Lemma 1 and (i). After comparing both expressions, since \( \text{char } R \neq 2 \), we get
\[
\{(uv)' - u'v - uv' \}(uv - vu) = 0, \quad \text{i.e.,} \quad u^c[u, v] = 0 \text{ for all } u, v \in U.
\]

**Lemma 4.** If \( U \not\subseteq Z \) is a Lie ideal of \( R \), then \( [u, v]^c = 0 \) for all \( u, v \in U \).

**Proof.** Replace \( w \) by \( 2vu \) in Lemma 2 and continue by the same procedure as in Lemma 3 to get \( [u, v]^c = 0 \). But in view of condition (ii), \( u^c + v^c = 0 \), i.e., \( v^c = -u^c \). So we get the desired conclusion that \( [u, v]^c = 0 \) for all \( u, v \in U \).

**Lemma 5.** If \( U \not\subseteq Z \) is a Lie ideal of \( R \) and, for \( u \in U \), if \( u \in C_R(U) \), then \( u^c \in Z \).

**Proof.** By [2, Lemma 2], \( C_R(U) = Z \), so \( u \in Z \). From (ii), we have
\[
(2uv)' = (u'v + uv') + 2uv' \quad \text{for all } v \in U.
\]
Replacing \( v \) by \( vw + wc \) in the last equation, we get
\[
(2u(vw + wc))' = (u'(vw + wc) + (vw + wc)u') + 2u(vw + wc)'.
\]
Since \( u \in Z \), by Lemma 2 we get
\[
(2u(vw + wc))' = 2(uvw + wvu)'
\]
\[
= 2(u'vw + uv'w + uvw' + w'vu + wvu' + wvu')
\]
\[
= 2(u'vw + wvu') + 2(u'vw + vw' + w'v + v').
\]
Compare the two expressions for $(2u(vw + wv))'$ to obtain
$$u'(vw - wv) = (vw - wv)u'$$ for all $v, w \in U$,

i.e., $u' \in C_R([U, U]) = C_R(U)$ by [2, Lemma 3]. But, as above, $C_R(U) = Z$, so $u' \in Z$.

**Lemma 6.** If $U \not\subset Z$ is a Lie ideal of $R$ and, for $u, v \in U$, if $uv = vu$, then $u^c = 0$.

**Proof.** From Lemma 2, for all $w \in U$,
$$(uvw + wvu)' = u'vw + uv'w + uvw' + w'vu + wv'u + wvu'.$$

But by hypothesis $uv = vu$, so by (ii),
$$(uvw + wvu)' = (uv \cdot w + w \cdot uv)' = (uv)'w + (uv)w' + w'(uv) + w(uv)',$$

since $2uv \in U$ and char $R \neq 2$.

On comparing both expressions for $(uvw + wvu)'$, since $uv = vu$, we get
$$\{(uv)' - u'v - uv'\}w + w\{(vu)' - v'u - vu'\} = 0,$$

so $(u^c)w + w(v^c) = 0$. By (ii) $v^c = -u^c$, so $(u^c)w - w(u^c) = 0$ for all $w \in U$.

Then $u^c \in C_R(U) = Z$ by [2, Lemma 2]. So we conclude that for $u, v \in U$, if $uv = vu$ then $u^c \in Z$, i.e., $(uv)' - u'v - uv' \in Z$. Since $u^2 \in U$ and $u^2v = vu^2$, then
$$(u^2v)' - (u^2)'v - u^2v' \in Z,$$

so
$$(u^2v)' - (u'u + uu')v - u^2v' \in Z,$$

by (i). Again, as $2uv \in U$ and $u(2uv) = (2uv)u$, we get
$$(u(2uv))' - u'(2uv) - u(2uv)' \in Z,$$

i.e.,
$$(u^2v)' - u'(uv) - u(uv)' \in Z,$$

since char $R \neq 2$. Thus
$$u(u^c) = u\{(uv)' - u'v - uv'\} = \{(u^2v)' - (u'u + uu')v - u^2v'\}$$
$$- \{(u^2v)' - u'(uv) - u(uv)'\} \in Z.$$

If $u^c \neq 0$, since $R$ is prime and $u^c \in Z$, then we get $u \in Z$, so by Lemma 5, $u^c \in Z$.

Then by (ii), $u^c = 0$, a contradiction. Hence $u^c = 0$.

3. The main theorem. Now we are in position to prove the following theorem which extends a result of Herstein [1, Theorem 3.3].

**Theorem.** Let $R$ be a prime ring, char $R \neq 2$, and let $U$ be a Lie ideal of $R$ such that $u^2 \in U$ for all $u \in U$. If $'$ is an additive mapping of $R$ into itself such that $(u^2)' = u'u + uu'$ for all $u \in U$, then $(uv)' = u'v + uv'$ for all $u, v \in U$.

**Proof.** Linearizing Lemmas 3 and 4 on $v$, we get
$$u^c[u, w] + u^w[u, v] = 0,$$

i.e.,
$$u^c[u, w] = -u^w[u, v]$$
and
\[(2) \quad [u, w]u^v + [u, v]u^w = 0, \quad \text{i.e.,} \quad [u, w]u^v = -[u, v]u^w.\]

Multiplying by \([u, w_1]\) on the left-hand side of (1) and using (2) and (1) we get
\[(3) \quad [u, v]u^w[u, w_1] = -[u, w_1]u^w[u, v].\]

Let \(w_1 = 2w_1^2v_1^2\) in (3); since \(\text{char} R \neq 2\), we get
\[ [u, v]u^w[u, w_1]v_1 + [u, w_1]v_1u^w[u, v] = -[u, w_1]v_1u^w[u, v] - w_1[u, v_1]u^w[u, v]. \]

or
\[(4) \quad [u, v]u^w[u, w_1]v_1 + [u, w_1]v_1u^w[u, v] = -[u, v]u^w[w_1[u, v_1] - w_1[u, v_1]u^w[u, v]]. \]

Applying (1) and (2) to (3) we have
\[ [u, v]u^w[u, w_1] = [u, w_1]u^w[u, v], \]
\[ [u, w]u^v[u, w_1] = [u, w_1]u^v[u, v]. \]

and using these in (4) we obtain
\[ [u, w_1]u^v[u, w]v_1 + [u, w_1]v_1u^w[u, v] = -([u, v]u^w[w_1[u, v_1] - w_1[u, v_1]u^w[u, v]]. \]

In view of (1) and (2), the last equation gives
\[ [u, w_1]v_1u^w[u, v] = -([u, v]u^w[w_1[u, v_1] - w_1[u, v_1]u^w[u, v]]. \]

or
\[(5) \quad [u, w_1][u^w[u, w], v_1] = -([u, v]u^w[w_1[u, v_1] - w_1[u, v_1]u^w[u, v]]. \]

In (5), replace \(v_1\) by \(2v_1u_1\) and use (5). Since \(\text{char} R \neq 2\), we get
\[ [u, w_1]v_1u^w[u, v] = -([u, v]u^w[w_1[u, v_1] - w_1[u, v_1]u^w[u, v]]. \]

Replace \(v_1\) by \([u, w_1]\) in (6). Then
\[ [u, w_1][u^w[u, w], u_1] = -([u, v]u^w[w_1[u, v_1] - w_1[u, v_1]u^w[u, v]]. \]

Write \(v_1 = u_1\) in (5). Then
\[ [u, w_1][u^w[u, w], u_1] = -([u, v]u^w[w_1[u, v_1] - w_1[u, v_1]u^w[u, v]]. \]

and using this in the last equation we get
\[ -[u, w_1][u^w[u, v]u^w, w_1][u, w_1] = -([u, v]u^w[w_1[u, v_1] - w_1[u, v_1]u^w[u, v]]. \]

or
\[ ([u, v]u^w[w_1[u, v_1] - w_1[u, v_1]u^w[u, v] + [u, w_1][u^v[u, v]u^w, w_1][u, w_1][u, u_1] = 0. \]

Replace \(u_1\) by \(2u_2u_1\) in the last equation and use it to get
\[ ([u, v]u^w[w_1[u, v_1] - w_1[u, v_1]u^w[u, v] + [u, w_1][u^v[u, v]u^w, w_1][u, w_1][u, u_1] = 0. \]
If, for some \( u, u_1 \in U \), \( u^{u_1} \neq 0 \), then by Lemma 6 \( [u, u_1] \neq 0 \), so by [2, Lemma 4], we get

\[
[[u, v]u^w, w_1][u, w_1] - [u, w_1][[u, v]u^w, w_1] = 0.
\]

Write \( v_1 = w_1 \) in (5). Then in view of the last equation, we have

\[
[u, w_1][u^v[u, w], w_1] = -[[u, v]u^w, w_1][u, w_1] = -[u, w_1][[u, v]u^w, w_1],
\]
or

(7) \([u, w_1][u^v[u, w] + [u, v]u^w, w_1] = 0\).

Linearizing (7) on \( w_1 \) we have

(8) \([u, w_1][u^v[u, w] + [u, v]u^w, v_2] + [u, v_2][u^v[u, w] + [u, v]u^w, w_1] = 0\).

Now replace \( w_1 \) by \( 2uw_1 \) in (8). Since char \( R \neq 2 \) we get

\[
u[u, w_1][u^v[u, w] + [u, v]u^w, v_2] + [u, v_2][u^v[u, w] + [u, v]u^w, w_1] = 0.
\]

When \( w_1 = u \) in (8) then \([u, v_2][u^v[u, w] + [u, v]u^w, u] = 0\). Thus from the last equation we get

\[
u[u, w_1][u^v[u, w] + [u, v]u^w, v_2] + [u, v_2][u^v[u, w] + [u, v]u^w, w_1] = 0.
\]

But again in view of (8), the last equation reduces to

\[-u[u, v_2][u^v[u, w] + [u, v]u^w, w_1] + [u, v_2][u^v[u, w] + [u, v]u^w, w_1] = 0,
\]
or

(9) \([u, [u, v_2]][u^v[u, w] + [u, v]u^w, w_1] = 0\).

Replace \( w_1 \) by \( 2uw_2w_1 \) in (9) and use (9) to obtain

\([u, [u, v_2]]U[u^v[u, w] + [u, v]u^w, w_1] = 0\).

Then by [2, Lemma 4] either \([u, [u, v_2]] = 0 \) or \([u^v[u, w] + [u, v]u^w, w_1] = 0\). If \([u, [u, v_2]] = 0 \) for all \( v_2 \in U \), then by the Corollary of Theorem 1 [2] \([u, U] = 0\), i.e., \( u \in C_R(U) = Z \) by [2, Lemma 2], so by Lemma 6, \( u^{u_1} = 0 \), a contradiction. Hence

\([u^v[u, w] + [u, v]u^w, w_1] = 0 \) for all \( w_1 \in U \),

i.e., \( u^v[u, w] + [u, v]u^w \in C_R(U) = Z \). Thus, in view of (2), we get

(10) \( u^v[u, w] - [u, w]u^v \in Z \).

Commuting (10) with \( u^v \), since by (2) and Lemma 3, \( u^v[u, w]u^v = 0 \), then

(11) \( u^v(u^v[u, w] + [u, w]u^v) = 0 \).

Commuting (10) with \( [u, w] \), since by (1) and Lemma 4 \( [u, w]u^v[u, w] = 0 \), we get

(12) \( u^v[u, w]^2 + [u, w]u^v = 0 \).

Let us set \( \alpha = u^v[u, w] \) and \( \beta = [u, w]u^v \). By (2) and Lemma 3, we get \( u^v[u, w]u^v = -u^v[u, v]u^w = 0 \). Thus \( \alpha^2 = 0 \). Similarly, we can show \( \beta^2 = 0 \). In view of (12) and (11) we have

\[\alpha\beta = u^v[u, w]^2u^v = -[u, w]^2u^v = [u, w]u^v[u, w] = \beta\alpha.\]
Now
\[(\alpha - \beta)^3 = \alpha^3 + \alpha \beta^2 + \beta \alpha \beta + \beta^3 - \alpha^2 \beta - \alpha \beta^2 - \beta^3 = 0,\]
since \(\alpha^2 = \beta^2 = 0\) and \(\alpha \beta = \beta \alpha\). Since \(R\) is prime and by (10) \((\alpha - \beta) \in Z\), then \(\alpha - \beta = 0\), i.e., \(\alpha = \beta\). Thus we get
\[(13) \quad \u^v[u, w] = [u, w]u^v, \quad \text{i.e.,} \quad [u^v, [u, w]] = 0.\]
Let \(w = 2wu_3\) in (13). Then
\[
0 = [u^v, [u, 2wu_3]] = 2[u^v, [u, w]u_3 + w[u, u_3]] \\
= 2[u^v, [u, w]u_3] + 2[u^v, w[u, u_3]] \\
= 2[u^v, [u, w]]u_3 + 2[u, w][u^v, u_3] + 2[u^v, w][u, u_3] + 2w[u^v, [u, u_3]].
\]
By (13) the first and fourth terms are zero. Since \(\text{char } R \neq 2\), from above we get
\([u, w][u^v, u_3] + [u^v, w][u, u_3] = 0.\) Now replace \(w\) by \([u, w]\) and, in view of (13), we get
\([u, [u, w]][u^v, u_3] = 0.\) Replace \(u_3\) by \(2v_3u_3\) to get
\([u, [u, w]][u^v, u_3] = 0.\)
By [2, Lemma 4], either \([u, [u, w]] = 0\) or \([u^v, u_3] = 0.\) As above, we have seen that \([u, [u, w]] \neq 0\), therefore \([u^v, u_3] = 0\) and so \(u^v \in C_R(U) = Z.\) By Lemma 3, \(u^{u_3}[u, u_3] = 0.\) Since \(u^{u_3} (\neq 0) \in Z\) and \(R\) is prime, we get \([u, u_3] = 0.\) Therefore by Lemma 6, \(u^{u_3} = 0.\)
Hence for all \(u, v \in U, u^v = 0, \text{i.e.,} (uv)' = u'v + uv'.\)

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