QUOTIENT GROUPS OF FINITE GROUPS

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Abstract. Assume $H$ and $H_0$ are subgroups of the finite group $G$ with $H_0 \triangleleft H$.

Three theorems are presented which give criteria for the existence of a relative normal complement in $G$ of $H$ over $H_0$.

All groups in this paper are finite. Given a group $G$ with subgroups $H_0$, $H$ and $G$ such that $H_0 \triangleleft H$, we call $G_0$ a relative normal complement in $G$ of $H$ over $H_0$ if $G_0 \triangleleft G$, $G = G_0 H$ and $H_0 = G_0 \cap H$.

There are many theorems which assert that under various conditions on $H$ and $H_0$ there exists a normal complement in $G$ of $H$ over $H_0$. The purpose of this paper is to provide generalizations and analogues of some of these theorems. Before stating the results, we state and prove a theorem given by Dade in his course on group theory at the University of Illinois (1967). This theorem will be useful in the following proofs.

**Theorem 1 (Dade).** Assume $G$ is a group with subgroups $H$ and $H_0$ such that $H_0 \triangleleft H$, $(|G : H|, |H : H_0|) = 1$, and $H/H_0$ is abelian. If $x$ and $y$ are elements in $H$ which are $G$-conjugate, assume that $xH_0$ and $yH_0$ are conjugate in $H/H_0$. Then $G$ has a relative normal complement $G_0$ of $H$ over $H_0$.

**Proof.** If $v: G \to H/H_0$ is the transfer map, then the hypothesis yields that for $h \in H$, $v(h) = h^{|G : H|} H_0$. It follows that $v$ maps $H$ onto $H/H_0$ and it is immediate that $G_0 = \ker v$.

Theorem A below is a generalization of Dade’s Theorem. Theorem A also generalizes a theorem of Reynolds [4]. Theorem B below may be viewed as a generalization of a theorem of Brauer [1] in the case that $H/H_0$ is solvable. In particular, Theorem B is a generalization of Brauer’s Theorem when $|H/H_0|$ is odd. Corollary D may be viewed as an analogue of the theorems of Brauer [1] and Suzuki [5] when $\pi$ is a set of odd primes. If $G$ is a group, let $\pi(G)$ denote the set of primes dividing $|G|$. If $\pi$ is a set of primes, then the complementary set of primes will be denoted by $\pi'$. A group $G$ is a $\pi$-group if $\pi(G) = \pi$. If $x \in G$, then $x$ is a $\pi$-element if $\langle x \rangle$ is a $\pi$-group. Every element $x$ of $G$ has a unique decomposition $x = x_\pi x_{\pi'}$, where $x_\pi$ and $x_{\pi'}$ are powers of $x$. If $x$ and $y$ are elements of a subgroup $K$ of $G$, then $x$ and $y$ belong to the same $\pi$-section of $K$ if their $\pi$-parts $x_\pi$ and $y_\pi$ are $K$-conjugate. If $S$ is a subset of $G$, then $S^{G, \pi}$ denotes the union of all $\pi$-sections of $G$ which intersect $S$. 

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We will say that the configuration \((G, H, H_0)\) is satisfied if the following conditions hold:

(i) \(G\) contains subgroups \(H\) and \(H_0\) such that \(H_0 \triangleleft H\), and \([G : H] [H : H_0] = 1\).
(ii) If \(p \in \pi(H/H_0)\) and \(x\) and \(y\) are \(p\)-elements in \(H\) which are conjugate in \(G\), then \(xH_0\) and \(yH_0\) are conjugate in \(H/H_0\).

**Theorem A.** Assume that the configuration \((G, H, H_0)\) is satisfied and \(H/H_0\) is nilpotent. Then \(G\) contains a relative normal complement of \(H\) over \(H_0\).

**Theorem B.** Assume the configuration \((G, H, H_0)\) and the following conditions are satisfied:

(i) \(\pi(H/H_0)\) is solvable.
(ii) Every element \(g \in G\) of order \(p^aq^b\) whose \(p\)-part and \(q\)-part are each conjugate to elements of \(H - H_0\), is itself conjugate to an element of \(H - H_0\).

Then \(G\) has a relative normal complement of \(H\) over \(H_0\).

We note that the example \(G = S_5\), \(H = S_4\) and \(H_0 = 1\) shows that condition (ii) may not be removed from Theorem B.

A stronger condition than (ii) in Theorem B, but perhaps easier to understand is given in Corollary C.

**Corollary C.** Assume that the configuration \((G, H, H_0)\) is satisfied, \(H/H_0\) is solvable, and every \(\pi(H/H_0)\) element of \(G\) is conjugate to an element of \(H\). Then \(G\) has a relative normal complement of \(H\) over \(H_0\).

**Corollary D.** Let \(\pi\) be a set of odd primes. \(G\) contains a normal \(\pi\)-complement if and only if the following conditions are satisfied:

(i) \(G\) contains a Hall \(\pi\)-subgroup \(H\).
(ii) If \(p \in \pi\) and two \(p\)-elements of \(H\) are conjugate in \(G\), then they are conjugate in \(H\).
(iii) \(|(H - 1)^{G, \pi}| = |G : H||H - 1|\).

1. If \(K\) is a group and \(p\) a prime, let \(K_p\) denote a Sylow \(p\)-subgroup of \(K\). If \(\sigma\) is a set of primes, let \(|K|_\sigma = \prod_{p \in \sigma}|K_p|\).

**Lemma 1.** Assume the configuration \((G, H, H_0)\) is satisfied and \(H\) contains a normal subgroup \(H_1\) such that \(H \supseteq H_1 \supseteq H_0\). Then the configuration \((G, H, H_1)\) is satisfied. If \(G\) has a relative normal complement \(G_1\) of \(H\) over \(H_1\), and \(G_1\) has a relative normal complement \(G_0\) of \(H_1\) over \(H_0\), then \(G_0\) is a relative normal complement of \(H\) over \(H_0\).

**Proof.** Since \([H : H_1] [H : H_0], ([G : H] [H : H_1]) = 1\). Let \(p \in \pi(H/H_1)\) and assume that \(x\) and \(y\) are \(p\)-elements in \(H\) which are conjugate in \(G\). Then \(x^hH_0 = yH_0\) for some \(h \in H\). Since \(H_0 \subseteq H_1 \triangleleft H\), \(x^H_1 = yH_1\). Therefore, \((G, H, H_1)\) is satisfied.

Assume the subgroups \(G_1\) and \(G_0\) given in the lemma, exist. Let \(\pi = \pi(H/H_0)\), then \(O^\pi(G) \subseteq G_0\) since \(G/G_1\) and \(G_1/G_0\) are \(\pi\)-groups. Since \([G_0 : H_0]\) is a \(\pi\)' number, we have \(G_0 = H_0O^\pi(G)\). Therefore, \(G_0\) is normalized by \(H\). Now, \(G = HG_1 = HG_0\), yields \(G_0 \triangle G\). Further, \(H \cap G_0 = H_1 \cap G_0 = H_0\) so that \(G_0\) is a relative normal complement of \(H\) over \(H_0\).
Proof of Theorem A. We proceed by induction on \([H : H_0]\), noting that the theorem follows from Theorem 1 if \([H : H_0] = p\). Thus, we may assume \([H : H_0] \neq p\). Since \(H/H_0\) is nilpotent, \(H\) contains a proper normal subgroup \(H_1\) such that \(H_0 \not\subseteq H_1 \subseteq H\) and \(H_1/H_0 \subseteq Z(H/H_0)\). Lemma 1 implies that the configuration \((G, H, H_1)\) is satisfied. By induction, \(G\) has a relative normal complement \(G_1\) to \(H\) over \(H_1\). Clearly \(H_1/H_0\) is nilpotent. Let \(x, y \in H_1\) be \(G_1\)-conjugate \(p\)-elements for \(p \in \pi(H_1/H_0)\). Then \(xH_0\) and \(yH_0\) are conjugate in \(H/H_0\). But \(H_1/H_0 \subseteq Z(H/H_0)\) yields \(xH_0 = yH_0\). Therefore, \((G_1, H_1, H_0)\) satisfies the hypothesis of the theorem. Hence, \(G_1\) has a relative normal complement \(G_0\) of \(H_1\) over \(H_0\). Lemma 1 implies \(G_0\) is a relative normal complement of \(H\) over \(H_0\). 

Lemma 2. Assume the configuration \((G, H, H_0)\) is satisfied and \(H/H_0\) is solvable. Then \(G\) contains a normal subgroup \(H_1\) such that \(H_1 \supseteq H_0\) and \([H : H_1] = p_1\). Further, \(G\) has a relative normal complement \(G_1\) of \(H\) over \(H_1\).

Let \(x\) be a \(p\)-element in \(H_1\) where \(p \in \pi(H_1/H_0)\) and assume that \(x'\) is a \(p\)-element in \(xH_0\) such that \(x' \in H'\) and \(C_{H'/H_0}(x'H_0') \not\subseteq H'/H_0'\). If \(x^g \in H_1\) for \(g \in G_1\), then \(x^gH_0 = x^hH_0\) for some \(h \in H_1\).

Proof. Since \(H/H_0\) is solvable, there is a subgroup \(H_1\) as described in the lemma. Lemma 1 implies that \((G, H, H_1)\) is satisfied. Since \([H : H_1] = p_1\), the existence of \(G_1\) follows from Theorem 1.

If \(\bar{x} \in H_1\) let \(V_{\bar{x}} = \{z \mid z \in H, [\bar{x}, z] \in H_0\}\). It is direct to show that \(V_{\bar{x}}H_0 = C_{H/H_0}(\bar{x}H_0), V_{\bar{x}h_0} = V_{\bar{x}}\) for any \(h_0 \in H_0\) and \(V_{\bar{x}} = (V_{\bar{y}})^h\) if \(\bar{x} \in y^hH_0\) for \(y, h \in H\). Let \(x\) and \(x'\) be the elements described in the hypothesis. Now \(x' \in H'\) implies that \(x' = x_2'\) where \(x_2\) is a \(p\)-element in \(H\) whence \((V_{\bar{x}})' / H'_0 = C_{H'/H_0}(x'H_0')\). Thus, \(C_{H'/H_0}(x'H_0') \not\subseteq H'_1/H'_0\) yields \(V_{x_2} \not\subseteq H_1\). Since \(x'\) and \(x_2\) are \(G\)-conjugate \(p\)-elements in \(H\), \(x' = (x_2h_0)^h\) where \(h \in H\) and \(h_0 \in H_0\). It follows that \(V_x = V_{x'} = (V_{x_2h_0})^h = V_{x_2}'\). In particular, \(V_x \not\subseteq H_1\). Now \([H : H_1] = p_1\) and \(V_x \not\subseteq H_1\) imply that \(H = H_1V_x\). Accordingly, \([H : V_x] = [H_1 : V_x \cap H_1]\). Since \([H : V_x]\) is the number of conjugates of \(xH_0\) in \(H/H_0\) and \([H_1 : V_x \cap H_1]\) is the number of conjugates of \(xH_0\) in \(H_1/H_0\), we see that \(\{x^kH_0\mid h \in H\} = \{x^kH_0\mid h \in H_1\}\).

If \(x^g \in H_1\) for some \(g \in G_1\), then \(x^gH_0 = x^hH_0\) for some \(h \in H\). It follows from the previous paragraph that \(x^hH_0 = x^hH_0\) for \(h \in H_1\).

Lemma 3. Assume the configuration \((G, H, H_0)\) is satisfied and \(H/H_0\) is solvable. Let \(G_1, H_1\) be the subgroups given in Lemma 2. Then \((G_1, H_1, H_0)\) is satisfied and \(H_1/H_0\) is solvable.

Proof. Clearly \(H_1/H_0\) is solvable. Further, \(G = G_1H\) and \(G_1 \cap H = H_1\) imply \([G_1 : H_1] = [G : H]\). Hence, \(([G_1 : H_1], [H_1 : H_0]) = 1\). As in Lemma 2, let \(p_1 = [H : H_1]\).

Let \(x\) be a \(p\)-element in \(H_1\) where \(p \in \pi(H_1/H_0)\). In order to show that the configuration \((G_1, H_1, H_0)\) is satisfied, it is sufficient to show that if \(x^g \in H_1\) for \(g \in G_1\), then \(x^gH_0 = x^hH_0\) for some \(h' \in H_1\).

Let \(P\) be a Sylow \(p\)-subgroup of \(G_1\) which lies in \(H_1\) such that \(x \in P\). We first show that \(P_0 = H_0 \cap P\) is strongly closed in \(P\) with respect to \(G_1\). Indeed, let
y \in P_0^w \cap P$, then $y^w$ and $y \in P^w \leq H_i^w$. Hence, $y^wH_0^w = y^hH_0^w$ for some $h \in H$. Now $y \in H_0^w$ yields $y^wH_0^w = H_0^w$ so that $y^w \in H_0^w$. It follows that $y \in H_0$. Therefore, $y \in H_0 \cap P = P_0$ and $P_0$ is strongly closed in $P$ with respect to $G_1$.

Now, we may assume $g \in G_1$, $x \in P$, $x^g \in H_1$ and $x^gH_0 \neq x^hH_0$ for any $h' \in H_1$. Lemma 2 implies that if $x'$ is a $p$-element in $xH_0$ and $x' \in H'$, then $C_{H'/H_0}(x'H') \subseteq H_i'$. Thus, we may assume $x \in H_1 - H_0$. Since $x^g \in H_1$, there is $h \in H_1$ such that $x^gh \in P$. Since $P_0$ is strongly closed in $P$ with respect to $G_1$, Theorem B(a) of [2] implies there is a $k \in N_{G_1}(P_0)$ such that $x^ghP_0 = x^kP_0$.

The Frattini argument applied in $H$ yields $H = N_{P_0}(P_0)H_0$. Since the configuration $(G, H, H_0)$ is satisfied, there is an $h \in N_{P_0}(P_0)$ such that $x^ghH_0 = x^hH_0 = x^hH_0$. Thus, $\langle x, H_0 \rangle = \langle x^h, H_0 \rangle = \langle x, H_0 \rangle$. Since $k, h \in N_{G_1}(P_0)$, it follows that $\langle x, P_0 \rangle = \langle x^k, P_0 \rangle$ and $\langle x, P_0 \rangle$ are both Sylow $p$-subgroups of $\langle x, H_0 \rangle$. Hence, there is an $h_0 \in H_0$ such that $\langle x, P_0 \rangle = \langle x^k, H_0 \rangle = \langle x, P_0 \rangle$. Let $h = h_0h$, then $kh^{-1} \in N_{G_1}(x, P_0)$ and $x^h = x^{h_0}h = x[h, h_0]^h$. Noting that $[x, h_0]^h \in H_0$, we obtain $x^ghH_0 = x^hH_0 = x^hH_0$. Now $kh^{-1} \in N_{G_1}(x, P_0)$ implies that $(xP_0)^{kh^{-1}} = xP_0$.

We are assuming that $x^gh \neq x^hH_0$ for any $h' \in H_1$. Since $h_1$ and $k$ lie in $G_1$, $h \not\subseteq H_1$. In particular, $kh^{-1} \in G - G_1$. Thus $G_1 = \{H, H_1 \} = p$ implies that $(kh^{-1})_p \in (H - H_1)^{G_p}$. Let $v = (kh^{-1})_p$, then $v \in N\langle x, P_0 \rangle$ and $(xP_0)^v = xP_0$.

If $p = p_1$, then $T = \langle v, x, P_0 \rangle$ is a $p_1$ group. Hence $T = \langle v, x, P_0 \rangle \subseteq H'$ for some $r \in G$, $v \in H' - H'_1$ and $P_0 \subseteq H'_0$. Now $\langle v, x \rangle \not\subseteq P_0$ contradicts $C_{H'/H_0}(xH_0') \subseteq H'_1/H'_0$. Thus $p \neq p_1$, $v$ is a $p'$-element in $N_{G_1}(x, P_0)$, $v$ normalizes $P_0$, and $(xP_0)^v = xP_0$. It follows that $v$ centralizes some $p$-element $x'$ in $xP_0$. Using condition (ii) of Theorem B, we see that $x'v \in H'$ for some $r \in G$. Since $(p_1, p) = 1, x'$ and $v \in H'$. However, $v \in (H - H'_1)^r$ and $[x', v] = 1$ now imply $C_{H'/H_0}(x'H') \subseteq H'_1/H'_0$. Again this is a contradiction. Therefore, the configuration $(G_1, H_1, H_0)$ is satisfied. \qed

**Proof of Theorem B.** Let $G$ be a minimal counterexample to Theorem B, and let $(G_1, H_1, H_0)$ be the subsystem described in Lemma 2. We will show that $(G_1, H_1, H_0)$ satisfies the hypothesis of Theorem B. Lemma 3 implies that $(G_1, H_1, H_0)$ is satisfied and $H_1/H_0$ is solvable. Let $p$ and $q$ be distinct primes in $\pi(H_1/H_0)$, $x$ and $y$ are, respectively, $p$ and $q$-elements in $H_1 - H_0$ and $[x, y^g] = 1$ for some $g \in G_1$. Using $G = HG_1$ and condition (ii) of Theorem B we see that $xy^g \in (H - H_0)^r$ where $r \in G_1$. Hence, $xy^g \in (H - H_0)^r \cap G_1 = (H_1 - H_0)^r$. It follows that $(G_1, H_1, H_0)$ satisfies the hypothesis of Theorem B. By induction, $G_1$ has a relative normal complement $G_0$ of $H$ over $H_0$. Theorem B now follows from Lemma 1.

We note that Corollary C follows trivially from Theorem B.

**Proof of Corollary D.** If $G$ has a normal $\pi$-complement $G_0$, then the theorems of Brauer [I] and Suzuki [S] yield (i) and (ii). Moreover, these theorems imply that every nonidentity $\pi$-element in $G$ is conjugate to an element in $H - 1$. Since $G_0$ is the set of $\pi'$-elements in $G$, it follows that $(H - 1)^{G_1} = G - G_0$. In particular, $|H - 1|^{G_1} = |G : H|/|H - 1|$. Conversely, assume that conditions (i)-(iii) are satisfied. Let $G_0$ denote the set of $\pi'$-elements in $G$, then $|G_0| = k|G_0_{\pi'} = k|G : H|$ where $k$ is a positive integer.
Condition (iii) implies that $|G - (H - 1)^{G, \pi}| = |G_\pi|$. Since $G_0 \subseteq G - (H - 1)^{G, \pi}$, $k = 1$ and $G = G_0 \cup (H - 1)^{G, \pi}$ ($\cup$ denotes a disjoint union). Let $x$ and $y \in H - 1$ and assume $[x, y^g] = 1$ for some $g \in G$. Since $xy^g$ is a $\pi$-element, $xy^g \in G - G_0 = (H - 1)^{G, \pi}$. Thus, $(G, H, H_0)$ satisfies the hypothesis of Theorem B where $H_0 = 1$.

References