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QUOTIENT GROUPS OF FINITE GROUPS[1]

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ABSTRACT. Assume H and H_0 are subgroups of the finite group G with $H_0 \trianglelefteq H$. Three theorems are presented which give criteria for the existence of a relative normal complement in G of H over H_0.

All groups in this paper are finite. Given a group G with subgroups H_0, H and G such that $H_0 \trianglelefteq H$, we call G_0 a relative normal complement in G of H over H_0 if $G_0 \trianglelefteq G$, $G = G_0 H$ and $H_0 = G_0 \cap H$.

There are many theorems which assert that under various conditions on H and H_0 there exists a normal complement in G of H over H_0. The purpose of this paper is to provide generalizations and analogues of some of these theorems. Before stating the results, we state and prove a theorem given by Dade in his course on group theory at the University of Illinois (1967). This theorem will be useful in the following proofs.

THEOREM 1 (DADE). *Assume G is a group with subgroups H and H_0 such that $H_0 \trianglelefteq H$, $([G : H], [H : H_0]) = 1$, and H/H_0 is abelian. If x and y are elements in H which are G-conjugate, assume that xH_0 and yH_0 are conjugate in H/H_0. Then G has a relative normal complement G_0 of H over H_0.*

PROOF. If v: $G \to H/H_0$ is the transfer map, then the hypothesis yields that for $h \in H$, $v(h) = h^{[G:H]}H_0$. It follows that v maps H onto H/H_0 and it is immediate that $G_0 = \ker v$.

Theorem A below is a generalization of Dade's Theorem. Theorem A also generalizes a theorem of Reynolds [**4**]. Theorem B below may be viewed as a generalization of a theorem of Brauer [**1**] in the case that H/H_0 is solvable. In particular, Theorem B is a generalization of Brauer's Theorem when $|H/H_0|$ is odd. Corollary D may be viewed as an analogue of the theorems of Brauer [**1**] and Suzuki [**5**] when π is a set of odd primes. If G is a group, let $\pi(G)$ denote the set of primes dividing $|G|$. If π is a set of primes, then the complementary set of primes will be denoted by π'. A group G is a π-group if $\pi(G) = \pi$. If $x \in G$, then x is a π-element if $\langle x \rangle$ is a π-group. Every element x of G has a unique decomposition $x = x_\pi x_{\pi'} = x_{\pi'} x_\pi$ into a π-element x_π and a π'-element $x_{\pi'}$. Further, x_π and $x_{\pi'}$ are powers of x. If x and y are elements of a subgroup K of G, then x and y belong to the same π-section of K if their π-parts x_π and y_π are K-conjugate. If S is a subset of G, then $S^{G,\pi}$ denotes the union of all π-sections of G which intersect S.

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We will say that the configuration (G, H, H_0) is satisfied if the following conditions hold:

(i) G contains subgroups H and H_0 such that $H_0 \trianglelefteq H$, and $([G: H], [H: H_0]) = 1$.

(ii) If $p \in \pi(H/H_0)$ and x and y are p-elements in H which are conjugate in G, then xH_0 and yH_0 are conjugate in H/H_0.

THEOREM A. *Assume that the configuration* (G, H, H_0) *is satisfied and* H/H_0 *is nilpotent. Then* G *contains a relative normal complement of* H *over* H_0.

THEOREM B. *Assume the configuration* (G, H, H_0) *and the following conditions are satisfied*:

(i) H/H_0 *is solvable.*

(ii) *Every element* $g \in G$ *of order* $p^a q^b$ *whose p-part and q-part are each conjugate to elements of* $H - H_0$, *is itself conjugate to an element of* $H - H_0$.

Then G *has a relative normal complement of* H *over* H_0.

We note that the example $G = S_5$, $H = S_4$ and $H_0 = 1$ shows that condition (ii) may not be removed from Theorem B.

A stronger condition than (ii) in Theorem B, but perhaps easier to understand is given in Corollary C.

COROLLARY C. *Assume that the configuration* (G, H, H_0) *is satisfied,* H/H_0 *is solvable, and every* $\pi(H/H_0)$ *element of* G *is conjugate to an element of* H. *Then* G *has a relative normal complement of* H *over* H_0.

COROLLARY D. *Let* π *be a set of odd primes.* G *contains a normal* π*-complement if and only if the following conditions are satisfied*:

(i) G *contains a Hall* π*-subgroup* H.

(ii) *If* $p \in \pi$ *and two p-elements of* H *are conjugate in* G, *then they are conjugate in* H.

(iii) $|(H-1)^{G,\pi}| = [G:H]|H-1|$.

1. If K is a group and p a prime, let K_p denote a Sylow p-subgroup of K. If σ is a set of primes, let $|K|_\sigma = \prod_{p\in\sigma}|K_p|$.

LEMMA 1. *Assume the configuration* (G, H, H_0) *is satisfied and* H *contains a normal subgroup* H_1 *such that* $H \supseteq H_1 \supseteq H_0$. *Then the configuration* (G, H, H_1) *is satisfied. If* G *has a relative normal complement* G_1 *of* H *over* H_1, *and* G_1 *has a relative normal complement* G_0 *of* H_1 *over* H_0, *then* G_0 *is a relative normal complement of* H *over* H_0.

PROOF. Since $[H : H_1] | [H : H_0]$, $([G : H], [H : H_1]) = 1$. Let $p \in \pi(H/H_1)$ and assume that x and y are p-elements in H which are conjugate in G. Then $x^h H_0 = yH_0$ for some $h \in H$. Since $H_0 \subseteq H_1 \triangle H$, $x^h H_1 = yH_1$. Therefore, (G, H, H_1) is satisfied.

Assume the subgroups G_1 and G_0, given in the lemma, exist. Let $\pi = \pi(H/H_0)$, then $O^\pi(G) \subseteq G_0$ since G/G_1 and G_1/G_0 are π-groups. Since $[G_0 : H_0]$ is a π' number, we have $G_0 = H_0 O^\pi(G)$. Therefore, G_0 is normalized by H. Now, $G = HG_1 = HG_0$ yields $G_0 \triangle G$. Further, $H \cap G_0 = H_1 \cap G_0 = H_0$ so that G_0 is a relative normal complement of H over H_0.

PROOF OF THEOREM A. We proceed by induction on $[H : H_0]$, noting that the theorem follows from Theorem 1 if $[H : H_0] = p$. Thus, we may assume $[H : H_0] \neq p$. Since H/H_0 is nilpotent, H contains a proper normal subgroup H_1 such that $H_0 \not\subseteq H_1 \not\subseteq H$ and $H_1/H_0 \subseteq Z(H/H_0)$. Lemma 1 implies that the configuration (G, H, H_1) is satisfied. By induction, G has a relative normal complement G_1 to H over H_1. Clearly H_1/H_0 is nilpotent. Let $x, y \in H_1$ be G_1-conjugate p-elements for $p \in \pi(H_1/H_0)$. Then xH_0 and yH_0 are conjugate in H/H_0. But $H_1/H_0 \subseteq Z(H/H_0)$ yields $xH_0 = yH_0$. Therefore, (G_1, H_1, H_0) satisfies the hypothesis of the theorem. Hence, G_1 has a relative normal complement G_0 of H_1 over H_0. Lemma 1 implies G_0 is a relative normal complement of H over H_0. □

LEMMA 2. *Assume the configuration (G, H, H_0) is satisfied and H/H_0 is solvable. Then G contains a normal subgroup H_1 such that $H_1 \supseteq H_0$ and $[H : H_1] = p_1$. Further, G has a relative normal complement G_1 of H over H_1.*

Let x be a p-element in H_1 where $p \in \pi(H_1/H_0)$ and assume that x' is a p-element in xH_0 such that $x' \in H^r$ and $C_{H^r/H_0^r}(x'H_0^r) \not\subseteq H_1^r/H_0^r$. If $x^g \in H_1$ for $g \in G_1$, then $x^g H_0 = x^{h_1} H_0$ for some $h_1 \in H_1$.

PROOF. Since H/H_0 is solvable, there is a subgroup H_1 as described in the lemma. Lemma 1 implies that (G, H, H_1) is satisfied. Since $[H : H_1] = p_1$, the existence of G_1 follows from Theorem 1.

If $\tilde{x} \in H$, let $V_{\tilde{x}} = \{z \mid z \in H, [\tilde{x}, z] \in H_0\}$. It is direct to show that $V_{\tilde{x}}/H_0 = C_{H/H_0}(\tilde{x}H_0)$, $V_{\tilde{x}h_0} = V_{\tilde{x}}$ for any $h_0 \in H_0$ and $V_{\tilde{x}} = (V_y)^h$ if $\tilde{x} \in y^h H_0$ for $y, h \in H$. Let x and x' be the elements described in the hypothesis. Now $x' \in H^r$ implies that $x' = x_2^r$ where x_2 is a p-element in H whence $(V_{x_2})^r/H_0^r = C_{H^r/H_0^r}(x'H_0^r)$. Thus, $C_{H^r/H_0^r}(x'H_0^r) \not\subseteq H_1^r/H_0^r$ yields $V_{x_2} \not\subseteq H_1$. Since x' and x_2 are G-conjugate p-elements in H, $x' = (x_2 h_0)^h$ where $h \in H$ and $h_0 \in H_0$. It follows that $V_x = V_{x'} = (V_{x_2 h_0})^h = V_{x_2}^h$. In particular, $V_x \not\subseteq H_1$. Now $[H : H_1] = p_1$ and $V_x \not\subseteq H_1$ imply that $H = H_1 V_x$. Accordingly, $[H : V_x] = [H_1 : V_x \cap H_1]$. Since $[H : V_x]$ is the number of conjugates of xH_0 in H/H_0 and $[H_1 : V_x \cap H_1]$ is the number of conjugates of xH_0 in H_1/H_0, we see that $\{x^h H_0 \mid h \in H\} = \{x^h H_0 \mid h_1 \in H_0\}$.

If $x^g \in H_1$ for some $g \in G_1$, then $x^g H_0 = x^h H_0$ for some $h \in H$. It follows from the previous paragraph that $x^h H_0 = x^{h_1} H_0$ for $h_1 \in H_1$.

LEMMA 3. *Assume the configuration (G, H, H_0) is satisfied and H/H_0 is solvable. Let G_1, H_1 be the subgroups given in Lemma 2. Then (G_1, H_1, H_0) is satisfied and H_1/H_0 is solvable.*

PROOF. Clearly H_1/H_0 is solvable. Further, $G = G_1 H$ and $G_1 \cap H = H_1$ imply $[G_1 : H_1] = [G : H]$. Hence, $([G_1 : H_1], [H_1 : H_0]) = 1$. As in Lemma 2, let $p_1 = [H : H_1]$.

Let x be a p-element in H_1 where $p \in \pi(H_1/H_0)$. In order to show that the configuration (G_1, H_1, H_0) is satisfied, it is sufficient to show that if $x^g \in H_1$ for $g \in G_1$, then $x^g H_0 = x^{h'} H_0$ for some $h' \in H_1$.

Let P be a Sylow p-subgroup of G_1 which lies in H_1 such that $x \in P$. We first show that $P_0 = H_0 \cap P$ is strongly closed in P with respect to G_1. Indeed, let

$y \in P_0^w \cap P$, then y^w and $y \in P^w \leqslant H_1^w$. Hence, $y^w H_0^w = y^{h^w} H_0^w$ for some $h \in H$. Now $y \in H_0^w$ yields $y^w H_0^w = H_0^w$ so that $y^w \in H_0^w$. It follows that $y \in H_0$. Therefore, $y \in H_0 \cap P = P_0$ and P_0 is strongly closed in P with respect to G_1.

Now, we may assume $g \in G_1$, $x \in P$, $x^g \in H_1$ and $x^g H_0 \neq x^{h'} H_0$ for any $h' \in H_1$. Lemma 2 implies that if x' is a p-element in xH_0 and $x' \in H^r$, then $C_{H^r/H_0^r}(x'H^r) \subseteq H_1^r$. Thus, we may assume $x \in H_1 - H_0$. Since $x^g \in H_1$, there is $h_1 \in H_1$ such that $x^{gh_1} \in P$. Since P_0 is strongly closed in P with respect to G_1, Theorem B(a) of [**2**] implies there is a $k \in N_{G_1}(P_0)$ such that $x^{gh_1} P_0 = x^k P_0$.

The Frattini argument applied in H yields $H = N_H(P_0)H_0$. Since the configuration (G, H, H_0) is satisfied, there is an $h \in N_H(P_0)$ such that $x^{gh_1} H_0 = x^k H_0 = x^h H_0$. Thus, $\langle x^k, H_0 \rangle^{h^{-1}} = \langle x^{kh^{-1}}, H_0 \rangle = \langle x, H_0 \rangle$. Since $k, h \in N_G(P_0)$, it follows that $\langle x, P_0 \rangle^{kh^{-1}} = \langle x^{kh^{-1}}, P_0 \rangle$ and $\langle x, P_0 \rangle$ are both Sylow p-subgroups of $\langle x, H_0 \rangle$. Hence, there is an $h_0 \in H_0$ such that $\langle x, P_0 \rangle^{kh^{-1}h_0^{-1}} = \langle x, P_0 \rangle$. Let $\tilde{h} = h_0 h$, then $k\tilde{h}^{-1} \in N_G\langle x, P_0 \rangle$ and $x^{\tilde{h}} = x^{h_0 h} = x^h [x, h_0]^h$. Noting that $[x, h_0]^h \in H_0$, we obtain $x^{gh_1} H_0 = x^k H_0 = x^h H_0 = x^{\tilde{h}} H_0$. Now $k\tilde{h}^{-1} \in N_G\langle x, P_0 \rangle$ implies that $(xP_0)^{k\tilde{h}^{-1}} = xP_0$.

We are assuming that $x^g H_0 \neq x^{h'} H_0$ for any $h' \in H_1$. Since h_1 and k lie in G_1, $\tilde{h} \notin H_1$. In particular, $k\tilde{h}^{-1} \in G - G_1$. Thus $[G\colon G_1] = [H\colon H_1] = p_1$ implies that $(k\tilde{h}^{-1})_{p_1} \in (H - H_1)^{G, p_1}$. Let $v = (k\tilde{h}^{-1})_{p_1}$, then $v \in N\langle x, P_0 \rangle$ and $(xP_0)^v = xP_0$.

If $p = p_1$, then $T = \langle v, x, P_0 \rangle$ is a p_1 group. Hence $T = \langle v, x, P_0 \rangle \subseteq H^r$ for some $r \in G$, $v \in H^r - H_1^r$ and $P_0 \subseteq H_0^r$. Now $[v, x] \in P_0$ contradicts $C_{H^r/H_0^r}(xH_0^r) \subseteq H_1^r/H_0^r$. Thus $p \neq p_1$, v is a p'-element in $N_G\langle x, P_0 \rangle$, v normalizes P_0, and $(xP_0)^v = xP_0$. It follows that v centralizes some p-element x' in xP_0. Using condition (ii) of Theorem B, we see that $x'v \in H^r$ for some $r \in G$. Since $(p_1, p) = 1$, x' and $v \in H^r$. However, $v \in (H - H_1)^r$ and $[x', v] = 1$ now imply $C_{H^r/H_0^r}(x'H^r) \not\subseteq H_1^r/H_0^r$. Again this is a contradiction. Therefore, the configuration (G_1, H_1, H_0) is satisfied. $\square$

Proof of Theorem B. Let G be a minimal counterexample to Theorem B, and let (G_1, H_1, H_0) be the subsystem described in Lemma 2. We will show that (G_1, H_1, H_0) satisfies the hypothesis of Theorem B. Lemma 3 implies that (G_1, H_1, H_0) is satisfied and H_1/H_0 is solvable. Let p and q be distinct primes in $\pi(H_1/H_0)$, x and y are, respectively, p and q-elements in $H_1 - H_0$ and $[x, y^g] = 1$ for some $g \in G_1$. Using $G = HG_1$ and condition (ii) of Theorem B we see that $xy^g \in (H - H_0)^r$ where $r \in G_1$. Hence, $xy^g \in (H - H_0)^r \cap G_1 = (H_1 - H_0)^r$. It follows that (G_1, H_1, H_0) satisfies the hypothesis of Theorem B. By induction, G_1 has a relative normal complement G_0 of H over H_0. Theorem B now follows from Lemma 1.

We note that Corollary C follows trivially from Theorem B.

Proof of Corollary D. If G has a normal π-complement G_0, then the theorems of Brauer [**1**] and Suzuki [**5**] yield (i) and (ii). Moreover, these theorems imply that every nonidentity π-element in G is conjugate to an element in $H - 1$. Since G_0 is the set of π'-elements in G, it follows that $(H - 1)^{G, \pi} = G - G_0$. In particular, $|(H - 1)^{G, \pi}| = [G : H]|H - 1|$.

Conversely, assume that conditions (i)–(iii) are satisfied. Let G_0 denote the set of π'-elements in G, then $|G_0| = k|G|_{\pi'} = k[G : H]$ where k is a positive integer.

Condition (iii) implies that $\left|G-(H-1)^{G,\pi}\right|=|G|_{\pi'}$. Since $G_0 \subseteq G-(H-1)^{G,\pi}$, $k=1$ and $G=G_0 \dot{\cup} (H-1)^{G,\pi}$ ($\dot{\cup}$ denotes a disjoint union). Let x and $y \in H-1$ and assume $[x, y^g]=1$ for some $g \in G$. Since xy^g is a π-element, $xy^g \in G-G_0=(H-1)^{G,\pi}$. Thus, (G, H, H_0) satisfies the hypothesis of Theorem B where $H_0=1$.

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