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ON COUNTABLE COMPACTNESS AND SEQUENTIAL COMPACTNESS

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ABSTRACT. If a countably compact T_3 space X can be expressed as a union of less then c many first countable subspaces, then MA implies that X is sequentially compact. Also MA implies that every countably compact space of size $< c$ is sequentially compact. However, there is a model of ZFC in which $\omega_1 < c$ and there is a countably compact, separable T_2 space of size ω_1, which is not sequentially compact.

It is well known that every sequentially compact space is countably compact, but the reverse is false. Even compact spaces need not be sequentially compact. It is interesting to note that by adding some restrictions on the size of spaces, (countable) compactness of spaces is sometimes enough to guarantee sequential compactness. For instance, any compact space of size ω_1 is sequentially compact [**L**] (also see [**F**, **MS** and **W**]).

A natural question is whether countably compact spaces of size $< c$ are sequentially compact. The main result of this paper shows that it is undecidable in ZFC.

ω^* means $\beta\omega \setminus \omega$, and for any $P \subseteq \omega$, $P^* = (\mathrm{Cl}_{\beta\omega} P) \setminus P$ is a clopen subset of ω^*. Let X be a space with a dense subset $A = \{a_i: i < \omega\}$. Each $x \in X$ corresponds to a closed subset of ω^*; namely, $C_{x,A} = \bigcap \{\{i < \omega: a_i \in U\}^*: U$ is a neighborhood of $x\}$. Let $U_A^* = \{i: a_i \in U\}^*$.

LEMMA 1. (i) *A space X is countably compact iff $\bigcup \{C_{x,A}: x \in X\}$ is dense in ω^* for every infinite countable subset A of X.*

(ii) *A space X is sequentially compact iff for every infinite countable subset A, $\mathrm{Int}_{\omega^*} C_{x,A} \neq \varnothing$ for some $x \in X$.*

PROOF. (i) It suffices to note that for any infinite $P \subseteq \omega$ and $x \in X$, $C_{x,A} \cap P^* \neq \varnothing$ iff $x \in \mathrm{Cl}_x\{a_i: i \in P\}$.

(ii) It suffices to note that for any infinite $P \subseteq \omega$ and $x \in X$, $\{a_i: i \in P\}$ converges to x iff $P^* \subseteq C_{x,A}$.

For brevity, in the following we shall use C_x instead of $C_{x,A}$, if there is no ambiguity.

LEMMA 2 [**KS**]. (MA) *The union of less than c many nowhere dense subsets of ω^* is still nowhere dense in ω^*.*

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Thus we have

THEOREM 3. (MA) *Every countably compact space of size* $< c$ *is sequentially compact.*

Rudin and Kunen (see **[O]**) have shown that if a countably compact T_2 space can be expressed as a union of $\{X_n: n < \omega\}$, where for each $x \in X_n$, $\chi(x, X_n) = \omega$, then X is sequentially compact.

THEOREM 4. (MA) *If a countably compact* T_3 *space* X *can be expressed as a union of* $\{X_\alpha: \alpha < k\}$, *where* $k < c$ *and for each* $x \in X_\alpha$, $\chi(x, X_\alpha) < c$, *then* X *is sequentially compact.*

We need the following lemma.

LEMMA 5. *If* X *is a dense subset of a* T_3 *space* Y, *then for each* $x \in X$, $\chi(x, X) = \chi(x, Y)$.

PROOF OF THEOREM 4. Without loss of generality, let $A = \{a_i: i < \omega\}$ be a discrete countable dense subset of X with no subsets which are convergent sequences. Thus $\mathrm{Int}_{\omega^*} C_x = \varnothing$ for all $x \in X - A$ (by Lemma 1).

For each $\alpha < k$, let $D_\alpha = \bigcup\{C_x: x \in X_\alpha - A\}$. By Lemma 2, at least one of the D_α's is not nowhere dense. Let α_0 be the first of such α. By MA, there is an infinite $P_0 \subseteq \omega$ such that $P_0^* \cap \mathrm{Cl}_{\omega^*} D_\alpha = \varnothing$ for $\alpha < \alpha_0$ and $P_0^* \subseteq \mathrm{Cl}_{\omega^*} D_{\alpha_0}$. Since D_{α_0} is dense in P_0^* and P_0^* is clopen, there is a C_X with $C_X \cap P_0^* \neq \varnothing$ for some $x \in X_{\alpha_0} - A$. Then $x \in \mathrm{Cl}_X\{a_i: i \in P_0\}$.

By Lemma 5, we have $\chi(x, \mathrm{Cl}_X X_\alpha) < c$ for all $x \in X_\alpha$ and $\alpha < k$. By MA there is a $P \subseteq P_0$ with $P^* \subseteq \bigcap\{\{i < \omega: a_i \in U\}^*: U \in \mathfrak{U}\}$, where $\mathfrak{U}$ is a family of open neighborhoods of x such that $\mathfrak{U} \cap \mathrm{Cl}_X X_{\alpha_0}$ is a local base at x in $\mathrm{Cl}_X X_{\alpha_0}$ and $|\mathfrak{U}| < c$. Consider any $y \in X_{\alpha_0}$ with $y \neq x$. There exists an open neighborhood V of y and $U \in \mathfrak{U}$ such tht $\mathrm{Cl}_X U \cap \mathrm{Cl}_X V \cap \mathrm{Cl}_X X_{\alpha_0} = \varnothing$. If $|U \cap V \cap \{a_i: i \in P\}| = \omega$, take any $z \in \mathrm{Cl}_X\{a_i \in U \cap V: i \in P\}$ (i.e. $C_z \cap P^* \neq \varnothing$). Then $z \in \mathrm{Cl}_X X_{\alpha_0}$, but that contradicts $z \in \mathrm{Cl}_X U \cap \mathrm{Cl}_X V$. Therefore, $|U \cap V \cap \{a_i: i \in P\}| < \omega$ and $U_A^* \cap V_A^* \cap P^* = V_A^* \cap P^* = \varnothing$. Hence $P^* \cap C_y = \varnothing$. Besides, it is not hard to find an infinite $P_1 \subseteq P$ such that $P_1^* \cap C_x = \varnothing$, hence $P_1^* \cap D_{\alpha_0} = \varnothing$. Now, continue the above construction and inductively define an increasing sequence $\{\alpha_\delta: \delta < k\}$ and a decreasing sequence $\{P_\delta^* \subseteq \omega^*: \delta < k\}$ such that $P_\delta^* \cap \bigcup_{\beta < \alpha_\delta} D_\beta = \varnothing$. At the δth stage of induction, if δ is a limit ordinal, by MA we find $P_\delta \subseteq \omega$ with $P_\delta^* \subseteq \bigcup_{\nu < \delta} P_\nu^*$. Then we let $\alpha_\delta = \sup_{\nu < \delta} \alpha_\nu$. If $\delta = \nu + 1$ is a successor ordinal, P is defined as above, using $\mathrm{Cl}_X\{a_i: i \in P_\nu\}$ instead of X.

Finally, there is an infinite subset $P_k \subseteq \omega$ such that $P_k^* \cap \bigcup_{\alpha < k} D_\alpha = \varnothing$. This is a contradiction.

Now, we turn to the discussion of consistency of the negation of the statement in Theorem 3.

First we establish some lemmas. The following is due to K. Kunen **[BK]**.

LEMMA 6. *Let* M *be a countable transitive model with* $M \models GCH$. *There is an* ω_{ω_1}*-stage iterated forcing construction* $\langle \mathbf{P}_\alpha: \alpha < \omega_1 \rangle$ *of CCC posets (for definitions, see*

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[**B**, **K**]) *such that for each successor* $\alpha < \omega_1$, $M[G_\alpha] \vDash \mathrm{MA} + 2^\omega = \omega_\alpha$, *and* $M[G_\alpha] \vDash 2^\omega \leqslant \omega_{\alpha+1}$ *for each limit ordinal* α, *where the* G_α*'s are relevant* $\mathbf{P}_\alpha$*-generic filters over* M. *Let* $M[G_{\omega_1}] = \tilde{M}$.

PROOF. By induction on α we define $\mathbf{P}_\alpha$ and $\dot{\mathbf{Q}}_\alpha$ ($\alpha < \omega_1$) such that $\Vdash_{\mathbf{P}_\alpha} \dot{\mathbf{Q}}_\alpha$ is CCC and $|\dot{\mathbf{Q}}_\alpha| \leqslant \omega_{\alpha+1}$. Direct limits (finite support) are always taken at limit stages. In other cases let $\mathbf{P}_{\alpha+1} = \mathbf{P}_\alpha * \dot{\mathbf{Q}}_\alpha$. It is sufficient to note two facts which can be proved by induction:

(a) for any $\alpha < \omega_1$, $|\mathbf{P}_\alpha| \leqslant \omega_\alpha$ if α is successor, and $|\mathbf{P}_\alpha| \leqslant \omega_{\alpha+1}$ if α is limit;

(b) for any $\alpha < \omega_1$, $\Vdash_{\mathbf{P}_\alpha} \forall\lambda < \omega_{\alpha+1}, 2^\lambda \leqslant \omega_{\alpha+1}$.

In fact, (a) comes from the fact that $|\dot{\mathbf{Q}}_\alpha|^{M[G_\alpha]} \leqslant \omega_{\alpha+1}$ and Lemma 3.2 in [**B**]. Assertion (b) is true because $(2)^{\lambda M[G_\alpha]} \leqslant |\mathbf{P}_\alpha|^\lambda \leqslant \omega_\alpha^\lambda = \omega_{\alpha+1}$. By Theorem 6.3 in [**K**], or the remark after Theorem 3.4 in [**B**], there is a CCC poset $\mathbf{Q}_\alpha$ in $M[G_\alpha]$ such that $\Vdash_{\mathbf{P}_\alpha * \dot{\mathbf{Q}}_\alpha} \mathrm{MA} + 2^\omega = \omega_{\alpha+1}$ and $\Vdash_{\mathbf{P}_\alpha} |\dot{\mathbf{Q}}_\alpha| \leqslant \omega_{\alpha+1}$. Define $\mathbf{P}_{\alpha+1} = \mathbf{P}_{\alpha * \dot{\mathbf{Q}}_\alpha}$. Obviously, $\tilde{M} \vDash 2^\omega = \omega_{\omega_1}$.

LEMMA 7. *In the model* $\tilde{M}$, *there are clopen subsets* U_α ($\alpha < \omega_1$) *and* $W_{\alpha,\nu}$ *for* $\nu < \alpha$ *in* ω^*, *such that*:

(i) U_α, $W_{\alpha,\nu} \in M[G_{\alpha+2}]$ *for all* $\nu < \alpha < \omega_1$;

(ii) *for each* $\nu < \alpha$, $W_{\alpha,\nu} \subseteq \bigcap_{\nu \leqslant \beta < \alpha} W_{\beta,\nu} \cap (U_\nu \setminus \bigcup_{\nu < \beta \leqslant \alpha} U_\beta)$ *and* $W_{\alpha,\alpha} = U_\alpha$ (*it is equivalent to say that* $\forall \nu < \alpha$, $W_{\alpha,\nu} \subseteq \bigcap_{\nu \leqslant \beta < \alpha} W_{\beta,\nu} \setminus U_\alpha$);

(iii) *both* U_α *and* $\omega^* \setminus U_\alpha$ *meet every clopen subset of* ω^* *in* $M[G_{\alpha+1}]$;

(iv) *both* $W_{\alpha,\nu}$ *and* $\omega^* \setminus W_{\alpha,\nu}$ *meet every clopen subset of* $\bigcap_{\nu \leqslant \beta < \alpha} W_{\beta,\nu} \setminus U_\alpha$ *in* $M[G_{\alpha+1}]$;

(v) *for any clopen subset* $Q \subseteq \omega^*$, *there is* $\nu < \omega_1$ *such that for any* $\alpha > \nu$, $W_{\alpha,\nu} \cap Q \neq \varnothing \neq Q \setminus W_{\alpha,\nu}$.

PROOF. Suppose U_β and $W_{\beta,\nu}$ ($\nu \leqslant \beta$) are defined for all $\beta < \alpha$ and satisfy (i)–(iv). Work in $M[G_{\alpha+2}]$. Since $M[G_{\alpha+2}] \vDash \mathrm{MA} + c = \omega_{\alpha+2}$, there is a clopen subset U_α of $M[G_{\alpha+2}]$ such that U_α and $\omega^* \setminus U_\alpha$ both meet every clopen subset of ω^* in $M[G_{\alpha+1}]$. By the same reason, for any $\nu < \alpha$ there are clopen subsets R_ν of $M[G_{\alpha+2}]$ such that $\bigcup\{H;\ H \text{ is a clopen subset in } M[G_{\alpha+1}] \text{ and } H \subseteq \bigcap_{\nu \leqslant \beta < \alpha} W_{\beta,\nu}\} \subseteq R_\nu \subseteq \bigcap_{\nu \leqslant \beta < \alpha} W_{\beta,\nu}$. This is possible because the number of H's is less than $\omega_{\alpha+2}$ in $M[G_{\alpha+2}]$.

It is a simple application of $\mathrm{MA} + c = \omega_{\alpha+2}$ that, in $M[G_{\alpha+2}]$, there are clopen subsets $W_{\alpha\nu} \subseteq R_\nu \setminus U_\alpha$ for $\nu < \alpha$ such that both $W_{\alpha,\nu}$ and $\omega^* \setminus W_{\alpha,\nu}$ meet every clopen subset of $M[G_{\alpha+1}]$ which is contained in R_ν. Thus we have just completed our induction.

To check (v), consider an arbitrary clopen subset H of $\tilde{M}$ with $H \subseteq \omega^*$. Assume $H \in M[G_{\nu+1}]$; we can easily prove the following, which ends our proof.

Claim. For any $\alpha \geqslant \nu + 1$, $W_{\alpha,\nu+1} \cap H \neq \varnothing \neq H \setminus W_{\alpha,\nu+1}$.

The fact that $W_{\nu+1,\nu+1} \cap H \neq \varnothing \neq H \setminus W_{\nu+1,\nu+1}$ follows from the definition of $U_{\nu+1} = W_{\nu+1,\nu+1}$ and (iii). If $W_{\beta,\nu+1} \cap H \neq \varnothing \neq H \setminus W_{\beta,\nu+1}$ hold for all $\nu + 1 \leqslant \beta < \alpha$, there is an $H_1 \in M[G_{\alpha+1}]$ with $H_1 \subseteq \bigcap_{\nu+1 \leqslant \beta < \alpha} H \cap W_{\beta,\nu+1}$, whence $H_1 \subseteq R_{\nu+1}$ and $W_{\alpha,\nu+1} \cap H_1 \neq \varnothing \neq H_1 \setminus W_{\alpha,\nu+1}$.

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LEMMA 8. *In $\tilde{M}$ there is a maximal almost disjoint family (i.e. MADF) $\mathfrak{U}$ of size ω_1 on ω.*

PROOF. Suppose that a sequence $\{U_\beta: \beta < \alpha\}$ of clopen subsets of ω^* has been defined such that:

(i) $U_\beta \in M[G_{\beta+2}]$;

(ii) for any $\beta_1 < \beta_2$, $U_{\beta_1} \cap U_{\beta_2} = \varnothing$;

(iii) for any $\nu < \beta$ and $B \in \mathcal{P}(\omega)^{M[G_{\nu+1}]}$ with $B^* \cap (\bigcup_{\lambda<\beta} U_\lambda) = \varnothing$, $B^* \cap U_\beta \neq \varnothing$.

Let $\mathfrak{B} = \{B \in \mathcal{P}(\omega)^{M[G_\alpha]}; B^* \cap (\bigcup_{\beta<\alpha} U_\beta) = \varnothing\}$. Since $M[G_{\alpha+2}] \models |\mathfrak{B}| \leqslant \omega_{\alpha+1} < \omega_{\alpha+2}$ and MA $+ 2^\omega = \omega_{\alpha+2}$, there is a clopen subset U_α of ω^* in $M[G_{\alpha+2}]$ such that $\forall \beta < \alpha\ U_\alpha \cap U_\beta = \varnothing$ and $\forall B \in \mathfrak{B}\ U_\alpha \cap B \neq \varnothing$.

(i)–(iii) together imply that $\{U_\alpha: \alpha < \omega_1\}$ yields a MADF on ω.

LEMMA 9. *If there is a MADF of size ω_1 on ω, then there exists a MADF $\mathfrak{F}$ consisting of countable subsets on ω_1 with $|\mathfrak{F}| = \omega_1$. Besides, if ω_1 is a countable union of disjoint subsets A_n of size ω_1, there is an ADF $\mathfrak{F} \subseteq \mathfrak{B}$, where $\mathfrak{B} = \{B$; B is countable in ω_1 and $\forall n\ B \cap A_n$ is finite$\}$ and $\mathfrak{F}$ is maximal in $\mathfrak{B}$.*

The proof of Lemma 9 follows immediately from

LEMMA 10. *Suppose there is a MADF of size ω_1 on ω, and $\{W_i: i < \omega\}$ is a disjoint family of clopen subsets in ω^*. Then there is a disjoint family $\mathfrak{W}$ of clopen subsets in ω^* with $|\mathfrak{W}| = \omega_1$ such that $\forall W = \mathfrak{W}\ W \subseteq \omega^* \setminus \bigcup_{i<\omega} W_i$, and $\{W_i: i < \omega\} \cup \mathfrak{W}$ is maximal.*

PROOF. Let $\{U_\alpha: \alpha < \omega\}$ correspond to a MADF on ω, where the U_α's are clopen in ω^*. It is not hard to prove that there is a permutation h of ω such that $h^*(U_i) = W_i$ for $i < \omega$, where h^* is the automorphism of ω^* induced by h. For $\omega \leqslant \alpha < \omega_1$, let $W_\alpha = h^*(U_\alpha)$ and $\mathfrak{W} = \{W_\alpha: \omega \leqslant \alpha < \omega_1\}$. These sets are as desired.

We are now ready to attack our final result, which with Theorem 3 provides a complete answer to our question.

THEOREM 11. *It is consistent that $\omega_1 < c$ and there is a T_2 countably compact, nonsequentially compact space of size ω_1.*

PROOF. Work in the model $\tilde{M}$. Let $\{U_\nu: \nu < \omega_1\}$ and $\{W_{\delta,\nu}: \nu < \delta < \omega_1\}$ be the families mentioned in Lemma 10. Define $X_0 = \omega \cup \{\omega_1 + \alpha: \alpha < \omega_1\}$ and a topology τ_0 on X_0 as follows. Let ω be the set of all isolated points in X_0. For $\omega \leqslant \alpha$, let $\{\{\omega_1 + \alpha\} \cup B: B \subseteq \omega, B^*$ contains some $W_{\delta,\alpha}$ for $\alpha \leqslant \delta < \omega_1\}$ be the local base at the point $\{\omega_1 + \alpha\}$. Observe that τ_0 is T_2 by definition of $W_{\delta,\alpha}$'s. Since $\{W_{\delta,\alpha}: \alpha \leqslant \delta < \omega_1\}$ is a decreasing sequence for any fixed α, every point of X_0 is a p-point, i.e. for $x \in X_0$ and any countable neighborhoods G_i of x, there is an open subset G in X_0 such that $x \in G \subseteq \bigcap_{i<\omega} G_i$. It follows that X_0 satisfies

($*$) for any $x \in X_0$ and any countable subset A with $x \notin \overline{A}$, there are disjoint open subsets U and V such that $x \in U$ and $A \subseteq V$; and

($**$) for any two mutually separated countable subsets A and B (i.e. $\overline{A} \cap B = A \cap \overline{B} = \varnothing$), there are disjoint open subsets U and V such that $A \subseteq U$ and $B \subseteq V$.

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The verification of $(*)$ is easy. Actually, $(*)$ and $(**)$ are equivalent. It follows from the next fact.

Fact 1. For any Hausdorff space X, $(*)$ implies $(**)$.

Indeed, let $A = \{a_i: i < \omega\}$ and $B = \{b_i: i < \omega\}$ be separated. Since X is Hausdorff, there are open subsets G_m and $H_{m,n}$ ($m, n < \omega$) such that $a_m \in G_m$, $b_n \in H_{m,n}$ and $G_m \cap H_{m,n} = \varnothing$ and open subsets $G_{m,n}$ and H_n such that $a_m \in G_{m,n}$, $b_n \in H_n$ and $G_{m,n} \cap H_n = \varnothing$ for all $m, n < \omega$. Let $U_m = \bigcap_{n \leqslant m} G_{m,n} \cap G_m$ and $V_n = \bigcap_{m \leqslant n} H_{m,n} \cap H_n$. Let $U = \bigcup_{m<\omega} U_m$ and $V = \bigcup_{n<\omega} V_n$. U and V are as desired.

Note that under MA, $(*)$ even implies that if A is countable and $|B| < c$, and $\bar{A} \cap B = A \cap \bar{B} = \varnothing$, then A and B can be separated by disjoint open neighborhoods.

In the following, we are going to inductively build up X_α and τ_α for $\alpha < \omega_1$ such that:

(i) τ_α is T_2 and for $\alpha_1 < \alpha_2 \leqslant \alpha$, $(X_{\alpha_1}, \tau_{\alpha_1})$ is an open subspace of $(X_{\alpha_2}, \tau_{\alpha_2})$;

(ii) for $0 < \alpha' < \alpha$ and any infinite countable subset $A \subseteq \bigcup_{\beta \leqslant \alpha'} X_\beta$, there is a cluster point in $X_{\alpha'+1}$, and every point $x \in X_{\nu+1} \setminus \bigcup_{\beta \leqslant \nu} X_\beta$ is a limit of some sequence in $\bigcup_{\beta \leqslant \nu} X_\beta$, where $\nu + 1 \leqslant \alpha$;

(iii) $|X_\alpha| \leqslant \omega_1$ and $X_{\beta+1} \setminus X_\beta$ is discrete and of size ω_1 for $\beta < \alpha$;

(iv) X_α satisfies $(*)$ (hence $(**)$).

Assume that X_β, τ_β have been defined for all $\beta < \alpha$.

Case (a). α is a limit ordinal.

Let $X_\alpha = \bigcup_{\beta<\alpha} X_\beta$ and τ_α be generated by $\bigcup_{\beta<\alpha} \tau_\beta$. It is trivial to verify (i)–(iii). Check (iv). Consider the following

Fact 2. Let $x \in X_\alpha$ and any countable subset $A \subseteq X_\alpha$ with $x \notin \mathrm{Cl}_{\tau_\alpha} A$, where $\mathrm{Cl}_{\tau_\alpha} A$ is the relative closure of A in X_α, then there are disjoint open subsets U and V in X_α such that $x \in U$ and $A \subseteq V$.

Proof by induction. The conclusion of Fact 2 is true for $x \in X_0$, since each point $x \in X_0$ is a p-point and X_α is Hausdorff. Assume that the conclusion is true whenever $x \in X_\beta$ for $\beta < \nu < \alpha$. Now assume $x \in X_\nu$. Clearly, we can assume that $\nu = \beta + 1$.

Let x be the limit of $B = \{b_i: i < \omega\}$ by (ii), such that $B \cap \mathrm{Cl}_{\tau_\alpha} A = \varnothing$. Note that A and B are separated in X_α. Since X_α is Hausdorff, A and B are separated by disjoint open neighborhoods U and V by following the proof of Fact 1. Hence $\{x\} \in V$ and U, V are as desired.

Case (b). $\alpha = \beta + 2$ is a successor ordinal.

Let $X_{\beta+1} \setminus X_\beta = \{x_\delta: \delta < \omega_1\}$. Suppose $\mathfrak{D} = \{D_\xi: \xi < \omega_1\}$ is a MADF consisting of countable subsets on ω_1. Define $A_\xi = \{x_\delta: \delta \in D_\xi\}$ and rewrite A_ξ as $\{x_{i,\xi}: i < \omega\}$. Now, consider all ordinals ν with $\omega_1 \cdot (\alpha + 1) \leqslant \nu < \omega_1 \cdot (\alpha + 2)$. Enumerate them as $\{\nu_\xi: \xi < \omega_1\}$ and let $X_\alpha = X_{\beta+1} \cup \{\nu_\xi: \xi < \omega_1\}$. Define τ_α in such a way that the restriction of τ_α on $X_{\beta+1}$ is $\tau_{\beta+1}$. For any point ν_ξ, let $\{\nu_\xi\} \cup \bigcup\{G_i: n < i < \omega\}$ be the basic neighborhoods of $\{\nu_\xi\}$, where the G_i's are mutually disjoint open neighborhoods of $x_{i,\xi}$ ($i < \omega$), respectively. It is reasonable since $X_{\beta+1}$ is collectionwise Hausdorff restricted to countable families by $(*)$.

(ii) and (iii) are obvious and (i) follows from (iv). For (iv), consider x and A as

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above. Let us do the same thing as we did in Case (a). Suppose that the conclusion of Fact 2 is true for $x' \in X_{\beta'}$, where $\beta' < \nu \leqslant \alpha$ and $\nu = \delta + 1$ is a fixed ordinal. Let x be the limit of a sequence $B = \{b_i \colon i < \omega\} \subseteq X_\delta$ and $A = \{a_i \colon i < \omega\} \subseteq X_\alpha$ with $\mathrm{Cl}_{\tau_\alpha} A \cap (B \cup \{x\}) = \varnothing$. What we have to do is to show that A and B can be separated by disjoint open subsets. Without loss of generality, it suffices to consider two subcases.

Subcase (1). $A \subseteq X_{\beta+1}$. Since $\tau_{\beta+1}$ is Hausdorff, it is easy to see that $A \cap \mathrm{Cl}_{\tau_{\beta+1}} B = \varnothing$, hence A and B are separated in $X_{\beta+1}$, and they can be separated by disjoint open neighborhoods by (iv).

Subcase (2). $A \subseteq X_{\beta+2} \setminus X_{\beta+1}$ and each $a_m (m < \omega)$ is nonisolated in $X_{\beta+2}$. First observe that every $x \in X_0$ can be separated from A by disjoint open neighborhoods. According to the proof of Fact 1, we have to show that (I) each point $b_n \in B$ can be separated from A by disjoint open neighborhoods, and (II) each point $a_m \in A$ can be separated from B by disjoint open neighborhoods. Let A_{ξ_m} be the sequence in the MADF which converges to $a_m \in A$. Since each A_{ξ_m} is closed and discrete in $X_{\beta+1}$, $A_{\xi_m} \cap B$ is finite whenever $\nu = \alpha$ or $\nu < \alpha$. Thus B and $A_{\xi_m} \setminus B$ are separated in $X_{\beta+1}$. Now, (II) follows by using (**) in $X_{\beta+1}$ and (I) is true because it is just the induction hypothesis.

Case (c). $\alpha = \lambda + 1$, where λ is a limit ordinal.

Let $\lambda_n \uparrow \lambda$ and $\mathfrak{F} = \{F_\xi \colon \xi < \omega_1\}$ be the ADF consisting of countable subsets of X_λ such that each $F \in \mathfrak{F}$ only contains finitely many points of $X_{\lambda_{n+1}} \setminus X_{\lambda_n}$ for any n. Moreover, $\mathfrak{F}$ is maximal with respect to the above property. The existence of $\mathfrak{F}$ follows from Lemma 9. For each $\xi < \omega_1$, let $F_\xi = \{x_{i,\xi} \colon i < \omega\}$. Define $X_\alpha = X_\lambda \cup \{\nu \colon \omega \cdot \lambda < \nu < \omega \cdot (\lambda + 1)\}$. Define τ_α in such a way that the restriction of τ_α on X_λ is τ_λ and for any $\omega \cdot \lambda + \xi$ $(\xi < \omega_1)$, let $\{\omega \cdot \lambda + \xi\} \cup \bigcup \{G_i \colon n < i < \omega\}$ be the basic neighborhoods of the point $\omega \cdot \lambda + \xi$, where the G_i's are mutually disjoint open neighborhoods of $x_{i,\xi}$. The verification of (i), (ii) and (iv) is the same as we did. To prove (ii), it suffices to observe that if A is an infinite subset of X_λ without cluster points in X_λ, then it means that $A \cap (X_{\lambda_{n+1}} \setminus X_{\lambda_n})$ is finite for any n, and hence A has an infinite intersection with some $F_\xi \in \mathfrak{F}$. $\omega \cdot \lambda + \xi \in \mathrm{Cl}_{\tau_\alpha} A$.

Case (d). $\alpha = 1$.

The definition of X_1 is like Case (b), but instead of $X_{\beta+1} \setminus X_\beta$ we use $X_0 \setminus \omega$. Assertion (ii) follows from Lemma 7(v) and the definition of X_1.

Finally, it is easy to see that X is a T_2 countably compact space and $|X| = \omega_1$. We still have to prove

Claim. There are no convergent subsequences in ω.

PROOF. By induction. Clearly no points of $X_0 \setminus \omega$ could be limits of any subsequences of ω. If $A = \{n_i \colon i < \omega\} \subseteq \omega$, we have to show that no points in X are limits of A. Suppose it is true for all $\beta < \alpha$. Let $x \in X_\alpha \setminus \bigcup_{\beta < \alpha} X_\beta$ and $x = \lim_n z_n$, where $\{z_n \colon n < \omega\} \subseteq \bigcup_{\beta < \alpha} X_\beta$. For each n, there is a neighborhood G_n of z_n and an infinite subset $A_n \subseteq A$ such that $A_n \cap G_n = \varnothing$ by our induction hypotheses. Without loss of generality, $A_{n+1} \subseteq A_n$ and $A_n = \{k_{n,i} \colon i < \omega\}$ for all $n < \omega$. Moreover, we could assume that $k_{n,n} \in A_m$ for all $m > n$, hence $\{k_{n,n} \colon n < \omega\} \subseteq \bigcap_{n < \omega} A_n$. Thus $(\bigcup_{n < \omega} G_n) \cap \{k_{n,n} \colon n < \omega\} = \varnothing$ and x is not a limit point of A.

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REMARK 12. (i) Malyhin and Sapirovski proved that MA implies that every separable countably compact space of size $< c$ is compact. Is MA necessary there? The answer is yes. It can be shown by induction that ω is dense in X. Since any compact space of size ω_1 is sequentially compact **[F]**, it follows that we just constructed a T_2, separable, countably compact, noncompact space of size $< c$ in $\tilde{M}$.

(ii) It is easy to absolutely construct a T_3 countably compact, nonsequentially compact space of size c, e.g. see Example 3.10.19 in **[E]**, but no such spaces of size $< c$ exist even consistently.

(iii) The following are generalizations of the above results.

(a) (MA) If a T_2 countably compact space $X = \bigcup_{\alpha<k} X_\alpha$ $(k < c)$, where the X_α's are compact and sequential or $|X_\alpha| \leqslant c$, then X is sequentially compact.

(b) (MA) If a T_3 countably compact k-space $X = \bigcup_{\alpha<k} X_\alpha$ $(k < 2^c)$, where the X_α's are sequential, then X is sequentially compact.

(Assertion (b) is a simple corollary of a new result of Y. Tanaka.)

(c) (MA) If a T_2 separable countably compact space $x = \bigcup_{\alpha<k} X_\alpha$ $(k < c)$, where the X_α's are compact, then X is compact. (Compare with **[MS**, Corollary 1.4].)

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