PROPERTY C AND FINE HOMOTOPY EQUIVALENCES

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Abstract. We show that within the class of metric $\sigma$-compact spaces, proper fine homotopy equivalences preserve property $C$, which is a slight generalization of countable dimensionality. We also give an example of an open fine homotopy equivalence of a countable dimensional space onto a space containing the Hilbert cube.

1. Introduction. In this note we shall study the behaviour of some “dimensionality properties” of infinite-dimensional spaces under fine homotopy equivalences. Let us recall that a map $f : X \to Y$ is a fine homotopy equivalence if for every open cover $\mathcal{U}$ of $Y$ there exists a map $g : Y \to X$ such that $f \circ g$ is $\mathcal{U}$-homotopic to $\text{id}_Y$ and $g \circ f$ is $f^{-1}(\mathcal{U})$-homotopic to $\text{id}_X$. Let us mention that a closed map $f : X \to Y$ of an ANR $X$ onto an ANR $Y$ is a fine homotopy equivalence if: (a) all fibers of $f$ are contractible or (b) $f$ is a cell-like map, i.e. $f$ is a proper map with fibers of trivial shape (see [Hal and To]). We are interested in countable dimensional spaces (a space $X$ is countable dimensional if $X$ is a countable union of finite dimensional sets) and spaces having property $C$ (a metric space $X$ has property $C$, abbreviated $X \in C$, iff given any sequence $(\epsilon_n)_{n=1}^\infty$ of positive real numbers, there exists an open cover $\mathcal{U}$ of $X$ such that $\mathcal{U} = \bigcup_{n=1}^\infty \mathcal{U}_n$, where $\mathcal{U}_n$ is a pairwise disjoint family with $\text{diam}(U) < \epsilon_n$ for every $U \in \mathcal{U}_n$, $n \in \mathbb{N}$). Note that each metric, countable dimensional space has property $C$ and that a space containing a topological copy of the Hilbert cube $Q = [-1, 1]^\infty$ does not have property $C$ (for details see [Ha2]).

Because fine homotopy equivalences do not raise finite dimension, the following question was posed by D. Henderson and G. Kozlowski.

Question 1. Do cell-like maps, which are fine homotopy equivalences, preserve countable dimension?

In this note we will show that within the class of $\sigma$-compact spaces, proper fine homotopy equivalences preserve property $C$ and we give an example of an open fine homotopy equivalence $\alpha$ of the space $\sigma = \{(x_i) \in l_2 : x_i = 0 \text{ for all but finitely many } i\}$ onto the space $\Sigma = \{(x_i) \in l_2 : \sum_{i=1}^\infty (ix_i)^2 < \infty\}$. The map $\alpha : \sigma \to \Sigma$ "raises" dimension because $\sigma$ is countable dimensional but $\Sigma$ contains the Hilbert cube $Q$ and hence $\Sigma \notin C$. 

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2. The main result. In this section we formulate and prove our main result.

2.1. Theorem. Let $X$ be a $\sigma$-compact metric space with property $C$ and let $f: X \to Y$ be a proper fine homotopy equivalence of $X$ onto a metric space $Y$. Then $Y \in C$.

Proof. Because every space which is the countable union of compacta with property $C$, has property $C$ itself, it is enough to prove that each compact subset of $Y$ has property $C$. Let $A$ be a compact subset of $Y$ and let $B = f^{-1}(A)$. Let $\rho$ be an extension on $Y$ of a given metric on $A$. Define a compatible metric $d$ on $X$ by the formula $d(x_1, x_2) = \delta(x_1, x_2) + \rho(f(x_1), f(x_2))$, where $\delta$ is a compatible metric on $X$ and $x_1, x_2 \in X$. Observe that $\rho(f(x_1), f(x_2)) \leq d(x_1, x_2)$ for every $x_1, x_2 \in X$. Now choose a sequence $\{e_n\}^\infty_{n=1}$ of positive real numbers. Since $X \in C$, there is an open cover $\mathcal{V}$ of $X$ such that $\mathcal{V} = \bigcup_{n=1}^\infty \mathcal{V}_n$, where $\mathcal{V}_n$ is a pairwise disjoint family consisting of sets of diameter less than $e_n/3$. Because $B$ is compact we can choose a finite subfamily $\mathcal{V}'$ of $\mathcal{V}$ which covers $B$. Let $n_0 = \min(n \in \mathbb{N}: \mathcal{V}' \subseteq \bigcup_{m} \mathcal{V}_m)$ and let $W = \bigcup_{m < n} \mathcal{V}_m$. Then $f(W)$ is a neighborhood of $A$ in $Y$. Let $g: Y \to X$ be a map such that $\rho(f \circ g, id_Y) < \eta$, where $\eta = \frac{1}{3} + \min(e_1, e_2, \ldots, e_0)$, and $g(A) \subseteq W$. Then $\mathcal{U} = g^{-1}(\mathcal{V}')$ is a cover of $A$. We will show that the cover $\mathcal{U}$ has the properties required in the definition of property $C$ for the sequence $\{e_n\}^\infty_{n=1}$ and the metric $\rho$. To this end, first observe that $\mathcal{U} = \bigcup_{n=1}^{n_0} g^{-1}(\mathcal{V}_n \cap \mathcal{V}')$ and that $g^{-1}(\mathcal{V}_n \cap \mathcal{V}')$ is a pairwise disjoint family for $n = 1, 2, \ldots, n_0$. Let $V \subseteq \mathcal{V}_n \cap \mathcal{V}'$. We shall prove that $\text{diam}_\rho g^{-1}(V) < e_n$. Take $y_1, y_2 \in g^{-1}(V)$ and for $i = 1, 2$ let $x_i = g(y_i)$. Then

$$
\rho(y_1, y_2) \leq \rho(y_1, fg(y_1)) + \rho(fg(y_1), fg(y_2)) + \rho(y_2, fg(y_2))
$$

$$
< 2\eta/3 + \rho(f(x_1), f(x_2))
$$

$$
< 2\eta/3 + d(x_1, x_2) < 2\eta/3 + e_\eta/3 < e_n.
$$

We conclude that $\text{diam}_\rho g^{-1}(V) < e_n$. Observe that the cover $\mathcal{U}' = \mathcal{U} \cup \emptyset$ has the properties required in the definition of property $C$ for the sequence $\{e_n\}^\infty_{n=1}$ and the metric $\rho$.

Remark. In the proof of the theorem we used only the fact that the map is approximately right invertible, i.e. given an open cover $\mathcal{U}$ of $Y$ there exists a map $g: Y \to X$ such that $f \circ g$ is $\mathcal{U}$-close to $id_Y$.

G. Kozlowski [Ko] proved that a proper map $f: X \to Y$ between ANR's is a fine homotopy equivalence iff $f$ is a hereditary shape equivalence, i.e., $\text{Sh}(f^{-1}(A)) = \text{Sh}(A)$ for each compact set $A$ in $Y$. This result is used in the proof of the following

2.2. Corollary. Let $X$ be a $\sigma$-compact space with property $C$ and let $f: X \to Y$ be a hereditary shape equivalence. Then $Y$ has property $C$.

Proof. Without losing generality, we can assume that $X$ and $Y$ are compact. By the Freudenthal Expansion Theorem, see e.g. Borsuk [Bo], $X$ is the inverse limit of finite dimensional ANR's, say $X = \lim (X_n, f_n)$, with each $X_n$ an ANR. Let $M$ be the infinite mapping cylinder of the sequence $(X_n, f_n)$ with a copy of $X$ attached at its end. Then $M \in ANR$ and $M \in C$ (observe that we added a countable dimensional set to $X$). Let $\mathcal{V}_f = \{f^{-1}(y); y \in Y\} \cup \{\text{points}\}$, then $\mathcal{V}_f$ is a cell-like decomposition of $M$. Let $p_f: M \to M/\mathcal{V}_f$ be the quotient map. Because $f$ is a hereditary shape...
equivalence, \( M/\sigma_f \in \text{ANR} \) and \( p_f \) is a fine homotopy equivalence \([\text{Ko}]\). By Theorem 2.1, \( M/\sigma_f \in C \) and since \( Y \) embeds in \( M/\sigma_f \), \( Y \in C \). 

3. The example. In this section we construct an example of an open fine homotopy equivalence of \( \sigma \) onto \( \Sigma \).

3.1. Example. There exists a map \( \alpha: \sigma \to \Sigma \) such that:

1. \( \alpha \) is "onto",
2. \( \alpha \) is open,
3. point inverses of \( \alpha \) are homeomorphic to \( \sigma \),
4. \( \alpha \) is a fine homotopy equivalence.

**Proof.** Let \( \beta: K \to Q \) be an open map of the universal Menger curve \( K \) onto the Hilbert cube such that \( \beta^{-1}(q) \) is homeomorphic to \( K \) for each \( q \in Q \) (see \([\text{An}]\)). Let \( 2^K_f \) and \( 2^Q_f \) denote the hyperspaces of finite subsets of \( K \) and \( Q \), respectively. By \([\text{Cu}]\), \( 2^K_f \) is homeomorphic to \( \sigma \) and \( 2^Q_f \) is homeomorphic to \( \Sigma \). Let \( \alpha: 2^K_f \to 2^Q_f \) be the map defined by \( \alpha((k_1, k_2, \ldots, k_n)) = (\beta(k_1), \beta(k_2), \ldots, \beta(k_n)) \). Then \( \alpha \) satisfies (1)–(4). The conditions (1) and (2) are satisfied because the map \( \alpha \) is open and onto. We will check (3). Take distinct \( q_1, q_2, \ldots, q_n \in Q \), arbitrarily. Observe that

\[
\alpha^{-1}((q_1, q_2, \ldots, q_n)) = \{ A_1 \cup A_2 \cup \cdots \cup A_n: A_i \subseteq \beta^{-1}(q_i) \text{ is finite and nonempty} \}
\approx 2^K_f \times 2^K_f \times \cdots \times 2^K_f \approx \sigma^n \approx \sigma.
\]

It is not hard to check that \( \alpha \) is a \( UV^\infty \)-map, i.e., given \( y \in \Sigma \) and a neighborhood \( U \) of \( y \), there is a neighborhood \( V \subseteq U \) of \( y \) such that \( \alpha^{-1}(V) \) is contractible in \( \alpha^{-1}(U) \).

By \([\text{Ha1}]\), \( \alpha \) is a fine homotopy equivalence.

3. Questions. At the end of this note, we state some open problems.

**Question 2.** Let \( f: X \to Y \) be a closed fine homotopy equivalence such that \( X \in \text{ANR} \). If \( X \in C \), does it follow that \( Y \in C \)? This is true for \( \sigma \)-compact \( X \).

**Question 3.** Let \( f: W \to V \) be an affine map of a \( \sigma \)-compact convex subset \( W \subseteq l_2 \) onto a subset \( V \subseteq l_2 \). If \( W \in C \), does it follow that \( V \in C \)?

REFERENCES