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THE EQUIVALENCE OF ZERO SPAN AND ZERO SEMISPAN

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ABSTRACT. In this paper we introduce the idea of the symmetric span of a continuum, and show that continua with zero symmetric span are in class W. Continua with zero span have zero symmetric span, but the converse does not hold. We also show that if every subcontinuum of the continuum M is in class W then the span of M and the semispan of M agree. These results are then applied to show that the properties of having zero span and of having zero semispan are equivalent.

1. Introduction. The concept of the span of a metric space was introduced by A. Lelek in 1964 [7]. Since that time it has become an important concept in continuum theory, particularly in regard to chainability of continua. It is known, for instance [**7**, p. 210], that chainable continua have zero span. It is an open question as to whether or not continua which have zero span are chainable (see [**8**, Problem 1, p. 93; **5**, Question 7, p. 331]). In 1977 Lelek introduced the concept of semispan [**9**] by relaxing a condition in his original definition.

In this paper we introduce and investigate symmetric span, another idea related to span. In particular, we show that continua with zero symmetric span are in class W. We also investigate the relationship between semispan and span, and show that these measures agree for a class of continua including those having zero span.

2. Definitions. A *continuum* is a nondegenerate connected metric space. A *mapping* is a continuous function. Suppose M is a continuum with metric d. Denote the first and second projections of $M \times M$ onto M by π_1 and π_2, respectively. The *span* of M (respectively, the *semispan* of M), denoted by $\sigma(M)$ (respectively, $\sigma_0(M)$), is the least upper bound of the set of all numbers ε for which there exists a subcontinuum Z of $M \times M$ such that $\pi_2(Z) = \pi_1(Z)$ (respectively, $\pi_2(Z) \subset \pi_1(Z)$) and $d(x, y) \geqslant \varepsilon$ for each (x, y) in Z.

If $\mathcal{G}$ is a collection of sets, denote the union of the members of $\mathcal{G}$ by $\mathcal{G}^*$.

3. Symmetric span. As an example of a continuum with positive span, consider the unit circle in the complex plane, $S^1 = \{z \colon z \in C \text{ and } |z| = 1\}$. To see that $\sigma(S^1) > 0$

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let $Z = \{(z, -z): z \in S^1\}$. If (x, y) is in Z then $d(x, y) = 2$. To see that $\pi_1(Z) = \pi_2(Z)$ one need only observe that if (x, y) is in Z then (y, x) is in Z. Continua which have the last property of the continuum Z above are often constructed in studying the span of examples (see **[3]** for instance). This motivates the following definition.

If X is a set and $A \subset X \times X$, then $A^{-1} = \{(y, x): (x, y) \in A\}$, and A is said to be *symmetric* if $A = A^{-1}$. Suppose M is a continuum with metric d. The *symmetric span* of M, denoted by $s(M)$, is the least upper bound of the set of all numbers ε for which there exists a symmetric subcontinuum Z of $M \times M$ such that $d(x, y) \geq \varepsilon$ for each (x, y) in Z.

As observed before, if M is a continuum and Z is a subcontinuum of $M \times M$ such that $Z = Z^{-1}$, then $\pi_1(Z) = \pi_2(Z)$. A straightforward application of this observation proves the following theorem giving the relationship between the span and the symmetric span of a continuum.

THEOREM 1. *If M is a continuum, then $0 \leq s(M) \leq \sigma(M)$. Consequently if $\sigma(M) = 0$ then $s(M) = 0$.*

Howard Cook has shown that the dyadic solenoid has a property (actually, (3) of Theorem 2 below) that is equivalent to having symmetric span equal to zero. Thus the last implication in Theorem 1 cannot be reversed.

Some useful and easily established equivalences of the property of having zero symmetric span are given in the following theorem.

THEOREM 2. *Suppose M is a continuum. Then the following are equivalent.*

(1) $s(M) = 0$.

(2) *If Z is a subcontinuum of $M \times M$ such that $Z = Z^{-1}$ then Z intersects Δ_M.*

(3) *If Z is a subcontinuum of $M \times M$ and Z intersects Z^{-1}, then Z intersects Δ_M.*

THEOREM 3. *If M is a continuum and $s(M) = 0$ then M is atriodic and hereditarily unicoherent, and therefore hereditarily irreducible.*

PROOF. First suppose that M contains a triod. Then there are subcontinua H_1, H_2 and H_3 of M such that $H_i \cap H_j$ is a common continuum for all distinct pairs of i and j, and $H_i \not\subset H_j \cup H_k$ if $i \neq j$ and $i \neq k$. Let p_i be a point in $H_i - (H_1 \cap H_2 \cap H_3)$ for $i = 1, 2, 3$. Defining $C_i = \{p_i\} \times (H_j \cup H_k)$ for $\{i, j, k\} = \{1, 2, 3\}$, let $Z = \bigcup_{i=1}^{3}(C_i \cup C_i^{-1})$. Then Z is a subcontinuum of $M \times M$ and $Z = Z^{-1}$. However Z does not intersect Δ_M. This involves a contradiction with Theorem 2(2).

Now suppose that M contains a nonunicoherent subcontinuum. Then there are subcontinua H and K of M such that $H \cap K$ is not connected. Let p and q be points of $H \cap K$ which are in different components of $H \cap K$. Let V_p and V_q be mutually exclusive open sets containing p and q, respectively, such that $H \cap K \subset V_p \cup V_q$. Let T_H and T_K be the closures of the components of $V_p \cap H$ and $V_p \cap K$, respectively, which contain p. By **[10**, Theorem 50, p. 18] the continua T_H and T_K contain points r_H and r_K, respectively, which are not in V_p. Let

$$C = (\{q\} \times T_H) \cup (K \times \{r_H\}) \cup (\{r_K\} \times H) \cup (T_K \times \{q\})$$

and $Z = C \cup C^{-1}$. It is easily seen that $Z = Z^{-1}$ and that Z does not intersect Δ_M. Hence we have a contradiction again with Theorem 2(2).

Thus M is atriodic and hereditarily unicoherent. It follows from [**11**, Theorem 3.2, p. 456] that M is hereditarily irreducible.

4. Symmetric span and class W. A mapping f from a continuum X onto a continuum M is *weakly confluent* provided that for each subcontinuum H of M, there is a component K of $f^{-1}(H)$ such that $f(K) = H$. A continuum M is in *class W* provided that for each mapping f from a continuum onto M, f is weakly confluent.

THEOREM 4. *If M is a continuum and $s(M) = 0$ then M is in class W.*

PROOF. Suppose M is a continuum which is not in class W, and that $s(M) = 0$. Suppose X is a continuum and f is a mapping of X onto M which is not weakly confluent. Then there is a subcontinuum C of M such that no component of $f^{-1}(C)$ is mapped by f onto C. Since $s(M) = 0$, by Theorem 3, C is irreducible between some two points, p and q. Then no component K of $f^{-1}(C)$ has the property that both p and q are in $f(K)$.

Let $\mathcal{K}_p = \{K\colon K$ is component of $f^{-1}(C)$ and $p \in f(K)\}$, $\mathcal{K}_q = \{K\colon K$ is a component of $f^{-1}(C)$ and $q \in f(K)\}$, $A = \mathcal{K}_p^*$ and $B = \mathcal{K}_q^*$. It follows that A and B are mutually exclusive; otherwise a component of $f^{-1}(C)$ would contain both p and q.

We will now show that A and B are closed. Suppose $x_1, x_2, x_3, \ldots$ is a sequence of points in A with limit y. For each positive integer i, let K_i be the component of $f^{-1}(C)$ which contains x_i, and let z_i be a point of K_i such that $f(z_i) = p$. There exists a subsequence $K_{n(1)}, K_{n(2)}, K_{n(3)}, \ldots$ of $K_1, K_2, K_3, \ldots$ with a sequential limiting set, K_0, which is a continuum [**10**, Theorems 58, 59, pp. 23–24]. There exists a subsequence $z_{n(i(1))}, z_{n(i(2))}, z_{n(i(3))}, \ldots$ of $z_{n(1)}, z_{n(2)}, z_{n(3)}, \ldots$ which has a limit, z_0. Since f is continuous and $f(z_{n(i(j))}) = p$ for $j = 1, 2, 3, \ldots, f(z_0) = p$. Since $K_{n(1)}$, $K_{n(2)}, K_{n(3)}, \ldots$ has sequential limiting set K_0, z_0 is in K_0. Suppose t_0 is in K_0. Then there is a sequence of points $t_1, t_2, t_3, \ldots$, with limit t_0, such that t_j is in $K_{n(j)}$ for $j = 1, 2, 3, \ldots$. Since C is closed, f is continuous and $f(t_j)$ is in C for $j = 1, 2, 3, \ldots$, $f(t_0)$ is in C. Hence t_0 is in $f^{-1}(C)$. Therefore $K_0 \subset f^{-1}(C)$. Let H be the component of $f^{-1}(C)$ containing z_0. Since $f(z_0) = p$, H is in $\mathcal{K}_p$. Since z_0 is in K_0, $K_0 \subset f^{-1}(C)$, and K_0 is a continuum, it follows that $K_0 \subset H$, and hence y is in H. Thus y is in $A = \mathcal{K}_p^*$. Therefore A is closed. Likewise B is closed.

Now, since A and B are closed and mutually exclusive and $f^{-1}(C)$ is closed, there exist mutually exclusive open subsets of X, U_A and U_B such that $A \subset U_A$, $B \subset U_B$, and $f^{-1}(C) \subset U_A \cup U_B$. Let s and t be points of X such that $f(s) = p$ and $f(t) = q$ (s is in A, t is in B). Let

$$Y = ((X - U_A) \times \{s\}) \cup ((X - (U_A \cup U_B)) \times f^{-1}(C)) \cup ((X - U_B) \times \{t\}).$$

Note that Y is the union of three closed subsets of $X \times X$ and is therefore closed and hence compact. Let $Z = f \times f(Y)$. Since Y is compact and $f \times f$ is continuous, Z is compact.

We now show that (p, q) is in Z, (q, p) is in Z and that Z misses the diagonal, Δ_M of $M \times M$. Since s is in the A, s is in $X - U_B$, thus (t, s) is in Y and therefore $(f(s), f(t)) = (p, q)$ is in Z. Since t is in B, t is in $X - U_A$, thus (t, s) is in Y and hence $(f(t), f(s)) = (q, p)$ is in Z. Suppose (x, s) is in $(X - U_A) \times \{s\}$. Then x is not in A, hence $f(x) \neq p$. But $f(s) = p$ so $(f(x), f(s))$ is not in Δ_M. Likewise, if (x, t) is in $(X - U_B) \times \{t\}$ then $(f(x), f(t))$ is not in Δ_M. Suppose (x, y) is in $(X - (U_A \cup U_B)) \times f^{-1}(C)$. Then x is not in $f^{-1}(C)$ since $f^{-1}(C) \subset U_A \cup U_B$. Since y is in $f^{-1}(C)$, $f(x) \neq f(y)$. Therefore $(f(x), f(y))$ is not in Δ_M. Hence Z and Δ_M are mutually exclusive.

We now show that Z is a continuum by showing that it is connected. Suppose it is not connected. Then there exist mutually exclusive open subsets of $M \times M$, V_1 and V_2, such that $Z \subset V_1 \cup V_2$ and each of V_1 and V_2 contains a point of Z. Suppose x is in X. Then either x is in U_A, x is in U_B, or x is in $X - (U_A \cup U_B)$.

Suppose x is in U_A. Then (x, t) is the only point in Y with first coordinate x. Let $i(x)$ be the integer in $\{1,2\}$ such that $f \times f(x, t)$ is in $V_{i(x)}$. There exist open subsets G_x and R_x of X such that x is in G_x, t is in R_x, $G_x \subset U_A$ and $f \times f(G_x \times R_x) \subset V_{i(x)}$. If (x', w) is in $Y \cap (G_x \times X)$ then x' is in $G_x \subset U_A$ and hence $w = t$. Thus (x', w) is in $G_x \times R_x$. Hence there exist an open subset G_x of X and an integer $i(x)$ in $\{1,2\}$ such that x is in G_x and $f \times f((G_x \times X) \cap Y) \subset V_{i(x)}$. Likewise, for x in U_B there exist an open subset G_x of X and an integer $i(x)$ in $\{1,2\}$ such that x is in G_x and $f \times f((G_x \times X) \cap Y) \subset V_{i(x)}$.

Suppose x is in $X - (U_A \cup U_B)$. Since $f \times f(\{x\} \times f^{-1}(C)) = \{f(x)\} \times C$ is a continuum, $\{x\} \times f^{-1}(C) \subset (f \times f)^{-1}(V_i)$ for $i = 1$ or $i = 2$. Denote this integer by $i(x)$. By [**6**, Theorem 12, p. 142] there exist open subsets of X, G_x and R_x such that x is in G_x, $f^{-1}(C) \subset R_x$ and $G_x \times R_x \subset (f \times f)^{-1}(V_{i(x)})$. If (x', w) is in $(G_x \times X) \cap Y$ then either x' is in $X - (U_A \cup U_B)$ and w is in $f^{-1}(C)$ or $w = s$ or $w = t$. In the last two cases w is in $f^{-1}(C)$ also. Thus $(G_x \times X) \cap Y \subset G_x \times R_x$ and

$$f \times f((G_x \times X) \cap Y) \subset f \times f(G_x \times R_x) \subset V_{i(x)}.$$

In all of the three cases, x in U_A, x in U_B, and x in $X - (U_A \cup U_B)$, there is a w in X such that (x, w) is in Y. Hence, for each x in X, there is an integer $i(x)$ in $\{1,2\}$ and an open set G_x containing x such that $f \times f((G_x \times X) \cap Y) \subset V_{i(x)}$.

Since V_1 intersects Z, there is a point (x, w) in Y such that $f \times f(x, w)$ is in V_1. Therefore there is a point x in X such that $i(x) = 1$. Similarly, there is a point x in X such that $i(x) = 2$. Let

$$W_1 = \{G_x : i(x) = 1\}^* \quad \text{and} \quad W_2 = \{G_x : i(x) = 2\}^*.$$

The sets W_1 and W_2 are open.

Suppose r is in $W_1 \cap W_2$. Then there is a point x_1 in X such that r is in G_{x_1} and $i(x_1) = 1$ and a point x_2 in X such that r is in G_{x_2} and $i(x_2) = 2$. Let y be a point in X such that (r, y) is in Y. Now $f \times f(r, y)$ is in V_j since $f \times f((G_{x_j} \times X) \cap Y) \subset V_j$, for $j = 1, 2$. This is inconsistent with the fact that V_1 and V_2 are mutually exclusive. Hence W_1 and W_2 are mutually exclusive.

Thus $X \subset W_1 \cup W_2$ where W_1 and W_2 are mutually exclusive nonempty open subsets of X. This is a contradiction since X is a continuum. Therefore Z is a continuum. Since (p, q) is in Z and (q, p) is in Z and Z does not intersect Δ_M, we have a contradiction with Theorem 2(3). Therefore M is in class W.

5. A class of continua for which the span and semispan are equal. A. Lelek has given an example [**9**, Example 1.5, p. 37] of a continuum X such that $\sigma_0(X) = 1$ and $\sigma(X) = \frac{1}{2}$. The following theorem shows that under suitable hypothesis the span and semispan agree.

THEOREM 5. *If every subcontinuum of the continuum M is in class W, then* $\sigma(M) = \sigma_0(M)$.

PROOF. Suppose M is a continuum with metric d and that every subcontinuum of M is in class W. By [**9**, Proposition 1.1, p. 36], $\sigma(M) \leqslant \sigma_0(M)$. Suppose $\varepsilon > 0$ and $\sigma_0(M) \geqslant \varepsilon$. Let Z be a subcontinuum of $M \times M$ such that $\pi_2(Z) \subset \pi_1(Z)$ and $d(x, y) \geqslant \varepsilon$ for each point (x, y) in Z.

We choose inductively a sequence $K_0, K_1, K_2, \ldots$ of subcontinua of Z such that for each positive integer i,

$$K_i \subset K_{i-1} \tag{1}$$

and

$$\pi_1(K_i) = \pi_2(K_{i-1}). \tag{2}$$

Let $K_0 = Z$. Suppose that K_i has been chosen for $0 \leqslant i \leqslant n$ in such a way that (1) and (2) are satisfied. Since $K_n \subset K_{n-1}$, it follows that $\pi_2(K_n) \subset \pi_2(K_{n-1}) = \pi_1(K_n)$. Let ρ_n be the restriction of π_1 to K_n. By hypotheses, $\pi_1(K_n) = \rho_n(K_n)$ is in class W. Thus ρ_n is weakly confluent and there is a component T of $\rho_n^{-1}(\pi_2(K_n))$ such that $\pi_1(T) = \rho_n(T) = \pi_2(K_n)$. Choose one such T and denote it by K_{n+1}. Condition (2) then holds for $i = n + 1$. Condition (1) holds also because $K_{n+1} \subset \rho_n^{-1}(\pi_2(K_n)) \subset K_n$. Let $H = \bigcap_{i>0} K_i$. Then H is a continuum. Moreover

$$\pi_2(H) = \pi_2\Big(\bigcap_{i>0} K_i\Big) = \bigcap_{i>0} \pi_2(K_i) = \bigcap_{i>0} \pi_1(K_{i+1})$$
$$= \pi_1\Big(\bigcap_{i>0} K_{i+1}\Big) = \pi_1(H).$$

If (x, y) is in H then (x, y) is in Z and hence $d(x, y) \geqslant \varepsilon$. Therefore $\sigma(M) \geqslant \varepsilon$. Thus $\sigma(M) \geqslant \sigma_0(M)$, and therefore $\sigma(M) = \sigma_0(M)$.

6. The equivalence of zero span and zero semispan. We now apply Theorem 5 to prove the principal result of this paper.

THEOREM 6. *If M is a continuum then $\sigma(M) = 0$ if and only if $\sigma_0(M) = 0$.*

PROOF. Suppose M is a continuum. If $\sigma_0(M) = 0$ then $\sigma(M) = 0$ by [**9**, Proposition 1.1, p. 36].

Suppose $\sigma(M) = 0$. If H is a subcontinuum of M then $\sigma(H) = 0$ and hence $s(M) = 0$ by Theorem 1. Thus H is in class W by Theorem 4. Therefore, $\sigma_0(M) = \sigma(M) = 0$ by Theorem 5.

COROLLARY. *If M is a continuum and $\sigma(M) = 0$ then M has the incidence point property and M has the fixed point property.*

PROOF. Suppose M is a continuum, f is a mapping of X onto M and g is a mapping of X into M. Let $Z = \{(f(x), g(x)): x \in X\}$. Then Z is a subcontinuum of $M \times M$ and $\pi_2(Z) \subset M = \pi_1(Z)$. Since $\sigma(M) = 0$, $\sigma_0(M) = 0$ by Theorem 6 and thus Z intersects Δ_M, that is, there is a point x of X such that $f(x) = g(x)$. Hence M has the incidence point property. By letting $X = M$ and $f = \mathrm{id}_M$, we see that M also has the fixed point property.

7. A sum theorem for continua with zero span. Fugate [**2**, Theorem 1, p. 466] proved that if H and K are chainable continua which intersect, then $H \cup K$ is chainable if and only if $H \cup K$ is atriodic and unicoherent. Ingram [**4**, Theorem 2, p. 196] strengthened this to the following: If H and K are chainable continua which intersect, then $H \cup K$ is chainable if and only if $H \cup K$ is atriodic and $H \cap K$ is connected. In [**1**] Edwin Duda and James Kell prove that this statement remains true if chainable is replaced by zero semispan. We now apply Theorem 6 and the result by Duda and Kell to prove the following:

THEOREM 7. *If H and K are continua which intersect, $\sigma(H) = 0$, $\sigma(K) = 0$, $H \cup K$ is atriodic, and $H \cap K$ is connected, then $\sigma(H \cup K) = 0$.*

PROOF. Suppose that H and K are continua which intersect, $\sigma(H) = 0$, $\sigma(K) = 0$, $H \cup K$ is atriodic, and $H \cap K$ is connected. Then $\sigma_0(H) = 0$ and $\sigma_0(K) = 0$ by Theorem 6. Thus by [**1**, Theorem 3.2], it follows that $\sigma_0(H \cup K) = 0$. Therefore $\sigma(H \cup K) = 0$, again by Theorem 6.

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