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VIRTUAL PERMUTATIONS OF $Z[Z^n]$ COMPLEXES

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ABSTRACT. We extend the characterization of virtual permutation endomorphisms in the case where $\Pi_1(M) = Z^n$. We show that for endomorphisms of $Z[Z^n]$ complexes the appropriate eigenvalue condition is that all eigenvalues be roots of units of the group ring $Z[Z^n]$. Among these endomorphisms the virtual permutations are detected by K_0. The main application is in identifying Morse-Smale isotopy classes on these manifolds.

Morse-Smale diffeomorphisms are the simplest structurally stable dynamical systems. An important problem is to determine when a given isotopy class contains a Morse-Smale diffeomorphism. This problem was reduced by Shub and Sullivan to the algebraic problem of representing homology maps by special matrices called virtual permutations. In this note we report progress in the nonsimply connected case.

1. Introduction and definitions.

DEFINITION. A square integral matrix is called a *virtual permutation* if it has the form

$$\begin{pmatrix} p_1 & & & * \\ & p_2 & & \\ & & \ddots & \\ & 0 & & p_r \end{pmatrix}$$

where the p_i are signed permutation matrices or zero.

Let R be a ring and $E\colon C_* \to C_*$ an endomorphism of a free R complex. Two endomorphisms $E\colon C_* \to C_*$ and $F\colon D_* \to D_*$ are said to be chain homotopy equivalent if there exists a chain homotopy equivalence $h\colon C_* \to D_*$ such that $Fh \simeq hE$. Shub and Sullivan [**12**] proved that provided $\dim M \geq 6$ and $\Pi_1(M) = 0$, a diffeomorphism $f\colon M \to M$ is isotopic to a Morse-Smale diffeomorphism if and only if the endomorphism induced by f on the Z-chain complex of a handle decomposition of M is chain homotopy equivalent to a virtual permutation of a free Z-complex.

Franks and Shub [**5**] introduced an invariant in algebraic k-theory which detects when this chain condition is satisfied. A linear map is called *quasi-unipotent* if all its eigenvalues are roots of unity, and *quasi-idempotent* if the zero eigenvalue is also allowed. Let **QI** be the category whose objects are pairs (M, e) where M is a finitely generated Z-module and $e\colon M \to M$ is quasi-unipotent on $M/\operatorname{Torsion}(M)$. The obstruction group, called SSF by Bass [**1**] is the torsion subgroup of $K_0(\mathbf{QI})$. Given

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a quasi-idempotent chain map $E\colon C_* \to C_*$, let $\chi(E) = \sum_{i=0}^{\dim C_*}(-1)^i[E_i]$ where $[E_i]$ is the class in SSF. Shub and Franks prove that an endomorphism E is chain homotopy equivalent to a virtual permutation of a free Z-complex if and only if $E_*\colon H_*(C_*)\circlearrowleft$ is quasi-idempotent and $\chi(E) = 0$ [**5**, Theorem 3.3].

When the manifold is not simply connected one must use the chain complex in the universal cover $C_*(\tilde{M})$ over the group ring ZG, $G = \Pi_1(M)$. A virtual permutation over ZG has the same block form but the blocks p_i can have nonzero entries $\pm g$, $g \in G$. If f is isotopic to a Morse-Smale diffeomorphism the chain map $f_*\colon C_*(\tilde{M})\circlearrowleft$ is chain homotopy equivalent to a virtual permutation of a free ZG-complex; however the converse requires additional algebraic conditions to allow cancelling in the 1- and 2-handles [**8**].

A generalization of the theory of Franks and Shub for endomorphisms of ZG-complexes was introduced in [**9**]. Let **Mod** be the category whose objects are pairs (M, e) where M is a finitely generated ZG-module and $e\colon M \to M$ is an operator homomorphism associated with an automorphism of G.[3] A morphism $h\colon (M,e) \to (N,f)$ in **Mod** is a ZG-linear homomorphism $h\colon M \to N$ such that $fh = he$. By a *resolution* of an object of **Mod** we mean a (long) exact sequence

$$\begin{array}{ccccccccccc} 0 \to & N_k & \to & N_{k-1} & \to \cdots \to & N_0 & \to & M & \to 0 \\ & \downarrow {\scriptstyle f_k} & & \downarrow {\scriptstyle f_{k-1}} & & \downarrow {\scriptstyle f_0} & & \downarrow {\scriptstyle e} & \\ 0 \to & N_k & \to & N_{k-1} & \to \cdots \to & N_0 & \to & M & \to 0 \end{array}$$

such that all squares commute. Let $\mathbf{P} \subset \mathbf{Mod}$ be the subcategory of objects which have finite free resolutions such that all the endomorphisms f_i are represented by virtual permutations. Let $\mathbf{Q} \subset \mathbf{Mod}$ be the subcategory of objects which are direct summands of objects of $\mathbf{P}$, i.e. where there exist $(M', e') \in \mathbf{Mod}$ such that $(M \oplus M', e \oplus e') \in \mathbf{P}$. Let $\mathrm{SSF}(G) = K_0(\mathbf{Q})/K_0(\mathbf{P})$. If C_* is a free ZG-complex and each $(C_i, E_i) \in \mathbf{Q}$, the Euler characteristic $\chi(E) \in \mathrm{SSF}(G)$ is defined analogously. If in fact each (C_i, E_i) has a free inverse $(C_i', E_i') \bmod \mathbf{P}$, then it is not hard to show that (C_*, E) is chain homotopy equivalent to a virtual permutation of a free ZG-complex if and only if $\chi(E) = 0$ [**9**]. However, in [**9**] we were unable to directly identify the objects of $\mathbf{Q}$.

It follows from a key lemma of Franks and Shub [**5**, Proposition 2.7] that when $G = 0$ the quasi-idempotent endomorphisms are precisely the objects of $\mathbf{Q}$. If θ is a nonzero root of the characteristic polynomial of $e\colon M \to M$, i.e. a primitive kth root of unity, then $Z[\theta]$ is a Dedekind domain, so its ideal classes form a group. If e is irreducible and (M', e') is chosen to represent the inverse ideal class of (M, e), it follows that $(M \oplus M', e \oplus e')$ has a finite v.p. resolution.

When $G \neq 0$, ZG is not a Dedekind domain. However when $G = Z^n$, ZG is Noetherian and integrally closed. In this note we exploit these properties to identify the objects of $\mathbf{Q}$ in a special case. We thank Michael Shub and John Franks for helpful conversations. Special thanks also to Ken Kramer for an improvement in the proof of Lemma 2.

THEOREM. *Suppose $G = Z^n$, M is a free ZG-module and $e\colon M \to M$ is ZG-linear. Then $(M, e) \in \mathbf{Q}$ if and only if all eigenvalues of e are either zero or roots of elements $\pm g$, $g \in G$, i.e. roots of units of ZG.*

[3] In general, the induced map $f_*\colon C_*(\tilde{M})\circlearrowleft$ is Z-linear but not ZG-linear: if $A \in C_k(\tilde{M})$, $x \in G$, then $f_k(xA) = f_*(x)f_k(A)$.

2. Proof of Theorem. Assume $G = Z^n$. If A is a v.p. matrix over ZG then some power A^k has nonzero diagonal blocks of the form $\pm g\,\mathrm{Id}$, $g \in G$. Therefore, under the assumptions of the Theorem, if $(M, e) \in \mathbf{Q}$ then all nonzero eigenvalues of e are roots of group elements. The converse occupies the remainder of the paper.

DEFINITION. A matrix is called *companion-like* [**5**] if it has the block-diagonal form of a v.p. matrix but with blocks which are companion matrices.

It follows from [**5**, 1.5] and [**4**, 2.3] that a companion-like ZG-endomorphism satisfying our eigenvalue condition has a 2-step v.p. resolution. (The arguments of [**4**, 2.3] adapt easily for matrices over ZG.) Therefore it suffices to prove there exists a free ZG-module M' and $e'\colon M' \to M'$ satisfying our eigenvalue condition such that $e \oplus e'\colon (M \oplus M')\circlearrowleft$ is represented by a companion-like matrix.

Since G is free abelian, ZG is a commutative integral domain with only the trivial units $\pm g$, $g \in G$ [**11**, pp. 4, 592]. Recall that a ring is called *regular* if it has finite homological dimension. Following Kaplansky [**6**] we will call a ring *super-regular* if every localization at a prime ideal is regular.

LEMMA 1. *ZG is Noetherian, integrally closed, super-regular and a unique factorization domain.*

PROOF. These properties are preserved by localization and by adjoining an indeterminate [**3**, p. 623 and **6**, Theorem 171]. Now Z enjoys all four properties and if $\langle T\rangle$ is an infinite cyclic group, then the group ring $Z\langle T\rangle$ is obtained from the polynomial ring $Z[T]$ by localizing at the multiplicative set generated by T. The result follows by induction on the rank of G. □

Let $K = K(ZG)$ be the field of fractions of ZG and $V = M \otimes_{ZG} K$. Extend e to V by linearity. Let $f(T)$ be the characteristic polynomial of $e\colon V \to V$; it follows that $f(T)$ has a unique factorization

$$f(T) = f_1(T) \cdots f_r(T)$$

where the $f_i(T)$ are irreducible over $K[T]$. Therefore V has a filtration

$$0 \subset V_1 \subset \cdots \subset V_r = V$$

such that $e(V_i) \subset V_i$ and $e\colon (V_i/V_{i-1})\circlearrowleft$ has characteristic polynomial $f_i(T)$. Let $M_i = M \cap V_i$. The quotient modules M_i/M_{i-1} are torsion-free ZG-modules and ZG-lattices of the vector spaces V_i/V_{i-1} [**3**, p. 513].

We consider first the case where $f(T)$ is irreducible. We need the following fact: If $A \subset B$ are Noetherian, integrally closed domains such that B is a finitely generated A-module, then a finitely generated B-module is reflexive as a B-module if and only if it is reflexive as an A-module [**3**, Proposition 19, p. 537].

LEMMA 2. *Let N be a reflexive ZG-lattice of the K-vector space V and $e\colon N \to N$. Suppose e has characteristic polynomial $f(T)$, an irreducible factor of $T^m - g$, $g \in G$, and let θ be a root of $f(T)$. Then $ZG[\theta]$ is Noetherian and integrally closed and N is a projective $ZG[\theta]$-module.*

PROOF. Regard N as a $ZG[T]$-module for indeterminate T with multiplication by T given by $e\colon N \to N$. Since ZG is a U.F.D., using Gauss' lemma it follows that $\{g(T) \in ZG[T] \mid \text{for some } n \in N,\ g(T)n = 0\} = (f(T))$, the ideal of $ZG[T]$ generated by $f(T)$. Therefore N is a torsion free module over the ring $ZG[T]/(f(T)) \cong ZG[\theta]$.

Consider the $K[\theta]$ vector space $W = N \otimes_{ZG[\theta]} K[\theta]$. W has dimension one over $K[\theta]$ and so $W \cong K[\theta]$ as a $K[\theta]$ vector space, and N is a $ZG[\theta]$-lattice of W. Therefore N is isomorphic as a $ZG[\theta]$-module to a fractional $ZG[\theta]$-ideal I of $K[\theta]$ [**3**, p. 512, Ex. 1]. We show that N is reflexive as a $ZG[\theta]$-module; this will imply that I is a divisorial fractional ideal.

By [**7**, Theorem 16, p. 221] either: (i) $T^m - g$ is irreducible in $K[T]$; (ii) $g = \alpha^p$, where $\alpha \in K$ and $p \mid m$; or (iii) $g = -4\alpha^4$ and $4 \mid m$. Case (iii) cannot occur since $ZG \cong Z[Z^n]$ does not contain a 4th root of (-4). We consider the other cases separately.

Case (i) $f(T) = T^m - g$. Writing $G \cong Z^n$ multiplicatively, let G be the free abelian group generated by r_i, $1 \le i \le n$, and let $g = \prod r_i^{k_i}$. Since $T^m - g$ is irreducible we may as well assume $\theta = \prod r_i^{k_i/m}$ in $Q^n \supset Z^n$. If $\langle G, \theta\rangle$ is the subgroup of Q^n generated by G and θ, then $\langle G, \theta\rangle \cong G$. Therefore $ZG[\theta] \cong Z(\langle G, \theta\rangle) \cong ZG$. Since N is reflexive as a ZG-module it is reflexive as a $ZG[\theta]$-module, therefore I is a divisorial fractional ideal [**3**, p. 519, Ex. 2]. However $ZG[\theta]$ is also a unique factorization domain, in which case all divisorial fractional ideals are principal [**3**, p. 502] and therefore free modules.

Case (ii) $g = \alpha^p$. Since ZG is integrally closed $\alpha \in ZG$; since ZG has only trivial units, and a root of a unit is a unit, g is a pth power in G. Suppose that $g = h^s$ where $h \in G$, $s \mid m$ and s is maximal; let $m = sd$. Writing $h = \prod r_i^{e_i}$, the e_i are relatively prime. Since θ is a root of $f(T)$ we have $\theta^{sd} = g = h^s$. Thus $(\theta^d/h)^s = 1$ and $\rho = (\theta^d/h) \in ZG[\theta]$ is an s-root of unity.

We now introduce a change of variable. Choose λ_i, $1 \le i \le n$, so that $\sum_{i=1}^n \lambda_i e_i = 1$ and define $R_i = (\rho^{\lambda_i} r_i) \in ZG[\theta]$. Therefore

$$\theta^d = \rho h = \rho\left(\prod_{i=1}^n (\rho^{-\lambda_i} R_i)^{e_i}\right) = \rho^{1-\sum \lambda_i e_i} \prod R_i^{e_i} = \prod_{i=1}^n R_i^{e_i}.$$

Let $H = (\prod_{i=1}^n R_i^{e_i}) \in ZG[\theta]$. Let $\bar{G} \subset ZG[\theta]$ be the free abelian group generated by the R_i, $1 \le i \le n$. Then $ZG[\theta] = Z\bar{G}[\theta, \rho]$ and θ is a root of $(T^m - H) \in Z\bar{G}[T]$. However H is not a power in $\bar{G}$ since the e_i are relatively prime, and $H = -4\alpha^4$ with $\alpha \in K(Z\bar{G})$ would again imply the existence of a 4th root of (-4) in ZG. Therefore $T^m - H$ is irreducible in $K(Z\bar{G})[T]$.

We now follow Case (i). Let $\theta = \prod R_i^{e_i/d}$. Then

$$ZG[\theta] = Z\bar{G}[\theta, \rho] \cong Z[\rho]\bar{G}[\theta] \cong Z[\rho](\langle \bar{G}, \theta\rangle) \cong Z[\rho][Z^n].$$

Now $Z[\rho]$ is a Dedekind domain, so as in Lemma 1 it follows that $Z[\rho][Z^n]$ is Noetherian, integrally closed, and super-regular [**6**, p. 120]. It follows as before that I is divisorial; however $ZG[\theta]$ need not be a U.F.D. Now if $\mathcal{M}$ is a maximal ideal of $ZG[\theta]$, the localization $ZG[\theta]_{\mathcal{M}}$ is a regular local ring and hence a unique factorization domain [**6**, p. 130]. It follows that divisorial fractional ideals of $ZG[\theta]$ are invertible modules [**3**, Proposition 1, p. 503] and by [**2**, Proposition 7.5, p. 132] invertible modules are projective. □

We return to the general case. Let θ_i be a root of $f_i(T)$.

LEMMA 3. *M_i/M_{i-1} is a projective $ZG[\theta_i]$-module.*

PROOF. By Lemma 2 it suffices to show that M_i/M_{i-1} is a reflexive ZG-module. We start at the top and work down. Let $(M/M_{r-1})^{\#} = \bigcap_{p\in\mathcal{P}} (M/M_{r-1})_p$ where $\mathcal{P}$

is the set of prime ideals of height one of ZG, i.e. the reflexive cover of M/M_{r-1}. Choose N such that $M_{r-1} \subset N \subset V$ and $(M/M_{r-1})^{\#} = N/M_{r-1}$. By Lemma 2, N/M_{r-1} is a projective $ZG[\theta_r]$-module. $ZG[\theta_r]$ is a free ZG-module since θ_r is a root of the monic irreducible polynomial, $f_r(T) \in ZG[T]$. Hence N/M_{r-1} is a projective ZG-module.

Consider the short exact sequence

$$0 \to M_{r-1} \to N \to N/M_{r-1} \to 0.$$

Let $j\colon N/M_{r-1} \to N$ be a splitting map and $H = j(N/M_{r-1})$. Then $N \cong M_{r-1} \oplus H$. Now $M \subset N$ so

$$M = M \cap N = M \cap (M_{r-1} \oplus H) = M_{r-1} \oplus (M \cap H).$$

Thus $M/M_{r-1} \cong M \cap H$; since M is projective so is $M \cap H$. Thus M/M_{r-1} is a projective and hence reflexive ZG-module. The lemma follows by a finite induction. □

Proof of Theorem. Since M_i/M_{i-1} is a projective $ZG[\theta_i]$-module there exists N_i such that $(M_i/M_{i-1}) \oplus N_i \cong ZG[\theta_i]^{k_i}$. Let $f_i\colon N_i \to N_i$ be multiplication by θ_i. We have

$$\begin{array}{ccccc} M_i/M_{i-1} & \oplus & N_i & \to & ZG[\theta_i]^{k_i} \\ {\scriptstyle e_i}\downarrow & & \downarrow{\scriptstyle f_i} & & \downarrow{\scriptstyle x\theta_i} \\ M_i/M_{i-1} & \oplus & N_i & \to & ZG[\theta_i]^{k_i} \end{array}$$

Now $ZG[\theta_i]$ is a free ZG-module with basis $1, \theta, \ldots, \theta^{d_i-1}$, $d_i =$ degree (θ_i), and in this basis multiplication by θ_i is represented by the companion matrix of $f_i(T)$.

Let $N = \sum_{i=1}^{r} \bigoplus N_i$. $(M \oplus N, e \oplus f)$ is represented by a companion-like matrix. N need not be free but since M and $M \oplus N$ are free ZG-modules, N is stably free. Adding if necessary a free ZG-summand with the identity endomorphism we obtain the required (M', e') with M' free such that $(M \oplus M', e \oplus e')$ is companion-like. This completes the proof. □

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