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ORIENTABLE PRODUCTS IN $\mathcal{N}$

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ABSTRACT. It is well known that the square $W \times W$ of any smooth closed manifold W is cobordant to an orientable manifold. This note shows more specifically that a product $U \times V$ of smooth closed manifolds U and V is cobordant to an orientable manifold if and only if there is a smooth closed manifold W such that U and V are both products of W by orientable manifolds.

The following result will be established:

THEOREM. *Let U and V be any smooth closed manifolds. Then the product $U \times V$ is cobordant to an orientable manifold if and only if there are orientable manifolds U_0 and V_0 and a manifold W, not necessarily orientable, such that U and V are cobordant to $U_0 \times W$ and $V_0 \times W$, respectively.*

One can reformulate the preceding theorem as follows. Let $\mathcal{N}$ be the unoriented cobordism ring, and let $\mathcal{N}_0 \subset \mathcal{N}$ be the subring whose classes can be represented by smooth closed *orientable* manifolds; that is, $\mathcal{N}_0$ is the image of the oriented cobordism ring Ω under the forgetful homomorphism $\Omega \to \mathcal{N}$. One wants to show that $u \in \mathcal{N}$ and $v \in \mathcal{N}$ satisfy $uv \in \mathcal{N}_0$ if and only if there are elements $u_0 \in \mathcal{N}_0$, $v_0 \in \mathcal{N}_0$ and $w \in \mathcal{N}$ such that $u = u_0 w$ and $v = v_0 w$.

Here are the two steps of the proof. One first introduces the subring $\mathcal{W} \subset \mathcal{N}$ of Wall [**4**], and one uses the fact that both $\mathcal{W}$ and $\mathcal{N}$ are unique factorization domains to prove the preceding assertion with $\mathcal{W}$ substituted for $\mathcal{N}_0$. One then uses Wall's characterization of $\mathcal{N}_0 \subset \mathcal{W}$ as the kernel of a derivation $\mathcal{W} \xrightarrow{\partial} \mathcal{W}$ to reinstate $\mathcal{N}_0$ in the result. The author thanks Professor Robert E. Stong for his contribution to the proof.

Recall from [**3**] that $\mathcal{N}$ is a $Z/2$ polynomial algebra $Z/2[x_n \mid n \neq 2^r - 1]$ with one generator $x_n \in \mathcal{N}$ in each degree $n \geq 0$ not of the form $2^r - 1$ for $r > 0$. In particular, $x_0 \in \mathcal{N}$ is the identity element, consisting of the class $[\{*\}]$ of the singleton manifold $\{*\}$; the generators $x_n \in \mathcal{N}$ for $n \geq 4$ are not uniquely defined, of course, since one can always replace x_n by $x_n +$ (decomposable elements) whenever $n \geq 4$. For notational convenience we write $x(2^s)$ in place of x_{2^s} to denote any generator of degree 2^s.

In [**4**] Wall chooses specific generators x_n of $\mathcal{N}$, $n \neq 2^r - 1$, and he introduces the $Z/2$ polynomial subalgebra $Z/2[x_n, x(2^s)^2 \mid n \neq 2^r - 1, n \neq 2^s]$ as a subring $\mathcal{W} \subset \mathcal{N}$. It is known that $\mathcal{W}$ consists of all classes in $\mathcal{N}$ represented by at least one manifold with an integral first Stiefel-Whitney class, and since $\mathcal{N}_0$ consists of all classes in $\mathcal{N}$ represented by at least one manifold with a vanishing first Stiefel-Whitney class, the inclusion $\mathcal{N}_0 \subset \mathcal{W}$ is clear. Since $\mathcal{W}$ and $\mathcal{N}$ are polynomial rings over a field $Z/2$

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they are both unique factorization domains; unique factorization is not available in $\mathcal{N}_0$ itself.

Here is the classical property of squares in $\mathcal{N}$, quoted earlier. The even-degree generators x_{2k} appearing in the following proof differ by decomposable elements from the even-degree generators used to describe $\mathcal{W}$; however, this substitution does not affect the result.

LEMMA 1. *For any $w \in \mathcal{N}$ one has $w^2 \in \mathcal{N}_0$; a fortiori $w^2 \in \mathcal{W}$.*

PROOF. It suffices to select generators x_n of $\mathcal{N}$ such that each square x_n^2 is represented by an orientable manifold; in fact, the classical choices of [**3** and **1**] have this property. Specifically, if n is an even number $2k \geq 0$ one can choose the cobordism class $[RP^{2k}]$ of the real projective space RP^{2k} to be a generator x_{2k}, as in [**3**], where $RP^0 = \{*\}$; however, according to [**4**] the square $RP^{2k} \times RP^{2k}$ is cobordant to the (canonically oriented) complex projective space CP^{2k}. For any odd $n \neq 2^r - 1$ one can choose the cobordism class $[P(n)]$ of the Dold manifold $P(n)$ to be a generator x_n, as in [**1**]; since each $P(n)$ is itself orientable the same is true of $P(n) \times P(n)$.

(In fact, since the squares x_n^2 of the preceding odd-dimensional generators x_n all lie in a proper subring of $\mathcal{N}_0$, the preceding proof shows that all squares w^2 lie in some fixed proper subring of $\mathcal{N}_0$; see [**2**], for example, for further details.)

LEMMA 2. *Let u and v be any nonzero elements of $\mathcal{N}$ such that $uv = v_0 \in \mathcal{W}$, where v_0 is necessarily nonzero. Then there is a nonzero $u_0 \in \mathcal{W}$ such that $v_0 u = u_0 v$.*

PROOF. If $u_0 = u^2 \in \mathcal{W}$ then $v_0 u = uvu = u^2 v = u_0 v$.

One can regard $\mathcal{N}$ as an algebra generated over $\mathcal{W}$ by the identity element $x_0 \in \mathcal{N}$ and the specific generators $x(2^s) \in \mathcal{N}$ of [**4**]; a fortiori one can regard $\mathcal{N}$ as a $\mathcal{W}$-module spanned by x_0 and all products $x(2^{s_1}) \cdots x(2^{s_q})$ such that $s_1 < \cdots < s_q$.

LEMMA 3. *$\mathcal{N}$ is the free $\mathcal{W}$-module spanned by x_0 and all products $x(2^{s_1}) \cdots x(2^{s_q})$ such that $s_1 < \cdots < s_q$.*

PROOF. Clearly there is a direct sum decomposition $\mathcal{N} = \mathcal{W} \oplus \mathcal{M}$ over $\mathcal{W}$, where the summand $\mathcal{W}$ consists of $\mathcal{W}$-multiples of x_0 and the summand $\mathcal{M}$ is spanned by the remaining generators. Given any $\mathcal{W}$-linear relation $\sum u(s_1, \ldots, s_q) x(2^{s_1}) \cdots x(2^{s_q}) = 0 \in \mathcal{M}$, one multiplies by any fixed $x(2^{s_1}) \cdots x(2^{s_q})$ to obtain a $\mathcal{W}$-linear relation in $\mathcal{W} \oplus \mathcal{M}$ for which the coefficient of x_0 is $u(s_1, \ldots, s_q) x(2^{s_1})^2 \cdots x(2^{s_q})^2 = 0 \in \mathcal{W}$. Since $\mathcal{W}$ is a unique factorization domain it follows that all coefficients $u(s_1, \ldots, s_q) \in \mathcal{W}$ in the given $\mathcal{W}$-linear relation vanish, hence that the generators $\{x(2^{s_1}) \cdots x(2^{s_q}) \mid s_1 < \cdots < s_q\}$ of the summand $\mathcal{M}$ are linearly independent.

LEMMA 4. *Let u and v be any nonzero homogeneous elements of $\mathcal{N}$ such that $uv \in \mathcal{W}$; then there are homogeneous elements $u_0 \in \mathcal{W}$, $v_0 \in \mathcal{W}$ and $w \in \mathcal{N}$ such that $u = u_0 w$ and $v = v_0 w$.*

PROOF. One forms the quotient field $\mathcal{F}$ of the unique factorization domain $\mathcal{W}$; the tensor product $\mathcal{F} \otimes \mathcal{N}$ over $\mathcal{W}$ is then a vector space over $\mathcal{F}$. According to Lemma 2 the nonzero elements $1 \otimes u \in \mathcal{F} \otimes \mathcal{N}$ and $1 \otimes v \in \mathcal{F} \otimes \mathcal{N}$ lie in the same one-dimensional subspace of $\mathcal{F} \otimes \mathcal{N}$, and Lemma 3 provides a unique basis element of the form $1 \otimes w$ with $w \in \mathcal{N}$ of least degree; trivially, $\deg w \leq \deg u$ and $\deg w \leq$

deg v. There is then an element $u_0/w_0 \in \mathcal{F}$ such that $1 \otimes u = u_0/w_0 \otimes w$, where one may as well assume that u_0 and w_0 have no factors in common. One squares this relation and applies Lemma 1 to conclude that

$$u^2 \otimes x_0 = u_0^2 w^2 / w_0^2 \otimes x_0 \in \mathcal{F} \otimes \mathcal{N},$$

hence $w_0^2 u^2 = u_0^2 w^2 \in \mathcal{W}$ by Lemma 3. Since u_0 and w_0 have no factors in common it follows that w_0^2 divides w^2 in the unique factorization domain $\mathcal{W}$, hence w_0 divides w in the unique factorization domain $\mathcal{N}$. However, w_0 itself belongs to $\mathcal{W}$, so it cannot be of positive degree without contradicting the minimality of deg w; hence w_0 is the identity element x_0, so $u = u_0 w$ for a unique $u_0 \in \mathcal{W}$. Similarly $v = v_0 w$ for a unique $v_0 \in \mathcal{W}$, which completes the proof.

The goal of the next few lemmas is to replace $\mathcal{W}$ by $\mathcal{N}_0$ in Lemma 4.

LEMMA 5. *Let $p_1, \ldots, p_s$ be distinct primes in the unique factorization domain $\mathcal{N}$, and let $r_1, \ldots, r_s$ be natural numbers such that $p_1^{r_1} \cdots p_s^{r_s} \in \mathcal{W}$; then for each $i = 1, \ldots, s$ either $p_i \in \mathcal{W}$ or r_i is even.*

PROOF. Set $u = p_1^{r_1}$ and $v = p_2^{r_2} \cdots p_s^{r_s}$ and observe that u and v have no common factors; hence $p_1^{r_1} = u = u_0 \in \mathcal{W}$ by Lemma 4. If r_1 is an odd number $2q+1$ then set $u = p_1$ and $v = p_1^{2q}$ and use Lemma 4 once again to conclude either that $p_1 \in \mathcal{W}$ or that $p_1^{2q-1} \in \mathcal{W}$; if p_1 were not in $\mathcal{W}$ one could then set $u = p_1$ and $v = p_1^{2q-2}$ in Lemma 4 to conclude either that $p_1 \in \mathcal{W}$ or that $p_1^{2q-3} \in \mathcal{W}$, and so forth until one could not escape the conclusion $p_1 \in \mathcal{W}$. A similar argument applies to each of the remaining prime powers $p_2^{r_2}, \ldots, p_s^{r_s}$.

For later reference we remark that Lemma 5 is in fact equivalent to Lemma 4; the proof of the converse assertion appears in disguise in Lemma 9.

LEMMA 6. *If u and v are nonzero elements of $\mathcal{N}$ such that $u \in \mathcal{N}_0$ and $uv \in \mathcal{N}_0$, then $v \in \mathcal{N}_0$.*

PROOF. See p. 308 of [**4**].

LEMMA 7. *Let $p_1, \ldots, p_s$ be distinct primes in the unique factorization domain $\mathcal{N}$ such that $p_1 \cdots p_s \in \mathcal{N}_0$; then each of $p_1, \ldots, p_s$ belongs to $\mathcal{N}_0$.*

PROOF. Since $\mathcal{N}_0 \subset \mathcal{W}$ Lemma 5 implies that each of $p_1, \ldots, p_s$ belongs to $\mathcal{W}$. If $p_1 \cdots p_{s-1} \in \mathcal{N}_0$ then Lemma 6 implies $p_s \in \mathcal{N}_0$. Hence to prove that $p_s \in \mathcal{N}_0$, for example, it remains only to consider the case that $p_1 \cdots p_{s-1} \notin \mathcal{N}_0$, where $p_1 \cdots p_{s-1} \in \mathcal{W}$. Wall [**4**] provides an exact sequence $\Omega \xrightarrow{r} \mathcal{W} \xrightarrow{\partial} \mathcal{W}$ in which r is the forgetful homomorphism from Ω onto $\mathcal{N}_0 \subset \mathcal{W}$ and ∂ is a derivation of degree -1 over $Z/2$; thus $\mathcal{N}_0 = \ker \partial$. (Wall uses the notation ∂_1 for ∂; the preceding exact sequence is used later in Wall's paper to construct the portion $\Omega \xrightarrow{r} \mathcal{W} \xrightarrow{\partial_3} \Omega$ of the exact triangle which determines Ω.) In any event, if $p_1 \cdots p_{s-1} \in \mathcal{W}$ and $(p_1 \cdots p_{s-1}) p_s \in \mathcal{W}$ satisfy $p_1 \cdots p_{s-1} \notin \mathcal{N}_0$ and $(p_1 \cdots p_{s-1}) p_s \in \mathcal{N}_0$, one has $\partial(p_1 \cdots p_{s-1}) \neq 0 \in \mathcal{W}$ and $\partial((p_1 \cdots p_{s-1}) p_s) = 0 \in \mathcal{W}$ for the derivation ∂. Hence

$$(\partial(p_1 \cdots p_{s-1})) p_s + (p_1 \cdots p_{s-1}) \partial p_s = \partial(p_1 \cdots p_s) = 0$$

for $\partial(p_1 \cdots p_{s-1}) \neq 0$, and, since p_s does not divide $p_1 \cdots p_{s-1}$, it follows that p_s divides ∂p_s. Since $\deg(\partial p_s) = (\deg\, p_s) - 1$, this implies $\partial p_s = 0$; that is, $p_s \in \ker \partial = \mathcal{N}_0$ as claimed.

The next lemma is an analog of Lemma 5.

LEMMA 8. *Let $p_1, \ldots, p_s$ be distinct primes in the unique factorization domain $\mathcal{N}$, and let $r_1, \ldots, r_s$ be natural numbers such that $p_1^{r_1} \cdots p_s^{r_s} \in \mathcal{N}_0$; then for each $i = 1, \ldots, s$ either $p_i \in \mathcal{N}_0$ or r_i is even.*

PROOF. Write $p_1^{r_1} \cdots p_s^{r_s} = uv$, where u contains all squares and v is of the form $p_1^{q_1} \cdots p_s^{q_s}$, where each of $q_1, \ldots, q_s$ is either 0 or 1. In case $q_i = 0$ the corresponding r_i is even, and v is the product of the remaining distinct primes among $p_1, \ldots, p_s$. Since $u \in \mathcal{N}_0$ by Lemma 1, it follows from Lemma 6 that $v \in \mathcal{N}_0$; hence, in case $q_i = 1$, Lemma 7 implies $p_i \in \mathcal{N}_0$.

LEMMA 9. *Let u and v be any nonzero homogeneous elements of $\mathcal{N}$ such that $uv \in \mathcal{N}_0$; then there are homogeneous elements $u_0 \in \mathcal{N}_0$, $v_0 \in \mathcal{N}_0$ and $w \in \mathcal{N}$ such that $u = u_0 w$ and $v = v_0 w$.*

PROOF. Let $p_1, \ldots, p_s$ be the distinct primes occurring in the unique factorizations of u and v, so that $u = p_1^{m_1} \cdots p_s^{m_s}$ and $v = p_1^{n_1} \cdots p_s^{n_s}$ for natural numbers $m_1, \ldots, m_s$, $n_1, \ldots, n_s$. The hypothesis $uv \in \mathcal{N}_0$ and Lemma 8 guarantee for each $i = 1, \ldots, s$ either that $p_i \in \mathcal{N}_0$ or that $m_i + n_i$ is even. In the latter case $|m_i - n_i|$ is also even, and it remains only to recall from Lemma 1 that $p_i^2 \in \mathcal{N}_0$ for any $p_i \in \mathcal{N}$.

This completes the proof of the Theorem itself since Lemma 9 is precisely the reformulated version described earlier.

We remark that one can substitute $\mathcal{W}$ for $\mathcal{N}_0$ in the proof of Lemma 9 to complete the proof that Lemmas 4 and 5 are equivalent, as promised earlier. Lemmas 8 and 9 are similarly equivalent.

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