

SHORTER NOTES

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AN ELEMENTARY PROOF OF THE RUSSO-DYE THEOREM

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Those familiar with the useful theorem proved below, first proved in [1], may appreciate the simplicity of this proof.

THEOREM. *Let A be a C^* -algebra with unit element 1 and unitary group U . Then the closed unit ball in A is the closed convex hull of U .*

PROOF. It suffices to show that if $x \in A_1$, the open unit ball of A , then $x \in \overline{\text{co}}(U)$, the closed convex hull of U . It is easy to reduce this to showing that for every $u \in U$, $y = (x + u)/2 \in \overline{\text{co}}(U)$: For then $U \subset 2\overline{\text{co}}(U) - x$, which is closed and convex, so $\overline{\text{co}}(U) \subset 2\overline{\text{co}}(U) - x$, or $(x + \overline{\text{co}}(U))/2 \subset \overline{\text{co}}(U)$; if $x_0 \in U$, and $x_{n+1} = (x + x_n)/2$, $x_n \in \overline{\text{co}}(U)$ and $x_n \rightarrow x$.

But note that $y = ((xu^{-1} + 1)/2)u$, so (since $\|xu^{-1}\| = \|x\| < 1$) $\|y\| < 1$, and y is invertible. Thus $y = v|y|$, with v unitary and $(y^*y)^{1/2} = |y| = (w + w^*)/2$, where $w = |y| + i(1 - |y|^2)^{1/2}$ is also unitary. This concludes the proof and indeed proves the stronger result: $A_1 - U \subset U + U$.

NOTE ADDED IN PROOF. C. K. Fong has pointed out that essentially the same argument proves also that $A_1 \subset \overline{\text{co}}(U)$: Extend slightly the segment from u through x to $x' \in A_1$, and apply the given argument to x' instead of x ; then $x'_n \in \overline{\text{co}}(U)$, and for large n , x lies between x'_n and u .

REFERENCES

1. B. Russo and H. A. Dye, *A note on unitary operators in C^* -algebras*, Duke Math. J. 33 (1966), 413-416.

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