ON THE FIELD OF A 2-BLOCK. II

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Abstract. Every real 2-block \( B \) of a finite metabelian group contains an irreducible character \( \theta \) such that \( \mathbb{Q}(B) = \mathbb{Q}(\theta) \).

Let \( B \) be a \( p \)-block of a finite metabelian group \( G \) and let \( \xi \) be a primitive \( |G| \)th root of unity over the rationals \( \mathbb{Q} \). If \( \tau \in \mathcal{D}(\mathbb{Q}(\xi)/\mathbb{Q}) \), the Galois group, it was shown in [3] that the set of all \( \chi^\tau \), where \( \chi \) is an irreducible ordinary character of \( B \), form all the irreducible ordinary characters of some block denoted by \( B' \). Let \( \mathcal{K}(B) = (\tau \in \mathcal{D}(\mathbb{Q}(\xi)/\mathbb{Q}) | B' = B) \) and \( Q(B) \) the subfield of \( \mathbb{Q}(\xi) \) fixed by \( \mathcal{K}(B) \). Assuming \( B \) is a 2-block, it was proved in [3] that \( Q(B) = Q(\theta) \) for some irreducible ordinary character \( \theta \in B \) if the 2-Sylow subgroup of \( G' \) is cyclic (\( G' \) abelian). Such an equality does not hold in general. However, in this paper we prove that if \( B \) is real then \( Q(B) = Q(\theta) \) for some irreducible \( \theta \in B \). In particular, this gives another aspect of real 2-blocks studied in Gow [4], where it was proved that every real 2-block of any finite group has an irreducible real character.

Theorem. Let \( B \) be a real 2-block of a finite metabelian group \( G \). Then \( B \) contains an irreducible 2-rational character \( \theta \) of height zero such that \( Q(B) = Q(\theta) \).

Proof. We use the results in [1 and 2]. Let \( P \) be the 2-Sylow subgroup of the commutator group \( G' \), \( G' = P \times G_1 \), and \( H = P \times \Lambda, \Lambda \subseteq G_1 \), such that \( G'/H \) is cyclic. For any subgroup \( L \) of \( G' \) let \( K(L) \supseteq G' \) be a subgroup of \( N(L) \), the normalizer of \( L \) in \( G \), such that \( K(L)/L \) is a maximal abelian subgroup of \( N(L)/L \). Fix \( K(\Lambda) \) and if \( \Lambda \subseteq L \subseteq H \), pick \( K(H) \supseteq K(L) \supseteq K(\Lambda) \). Let \( \sigma \) be a linear 2-modular representation of \( K(\Lambda) \), \( S = \ker \sigma \), and \( S \cap G' = H \). Let \( B(\sigma, H) \) be the collection of all representations \( T'G \) where \( T' \) is a linear (ordinary) representation of \( K(L) \), \( \ker T' \cap G' = L \), with the modular representation \( \overline{T'_{K(\Lambda)}} \) being \( G \)-conjugate to \( \sigma \) (notations are the same as in [2]), and \( L \) runs over all subgroups of \( G', G'/L \) cyclic, \( \Lambda \subseteq L \subseteq H \). Include in \( B(\sigma, H) \) the composition factors (and their Brauer characters) of the modular representations \( \overline{T'_{K(\Lambda)}} \), and the characters of \( T'G \). From [2, §4] \( B(\sigma, H) \) is a 2-block and every 2-block of \( G \) is given in this form.

Fix \( H \) and \( \sigma \) and assume \( B(\sigma, H) \) is real. If \( \chi \) is the character afforded by \( T'G \in B(\sigma, H) \), then its complex conjugate \( \chi^c \in B(\sigma, H) \). Thus \( \overline{T'_{K(\Lambda)}} \) and \( (T'^c)_{K(\Lambda)} \) are \( G \)-conjugate to \( \sigma \). Since \( T'^c(k) = T'(k^{-1}) \) for all \( k \in K(L) \supseteq K(\Lambda) \), it follows

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that \( \sigma' \), defined by \( \sigma'(f) = \sigma(f^{-1}) \) for all \( f \in K(\Lambda) \), is \( G \)-conjugate to \( \sigma \). Since \( \sigma'(f) = \sigma(f^{-1}) \) means \( \ker \sigma' = \ker \sigma \), there exists \( y \in N(S) \), \( y \) unique modulo the inertia group \( I(\sigma) \), such that \( \sigma(y^{-1}fy) = \sigma(f^{-1}) \) for all \( f \in K(\Lambda) \), or \( y \) inverts every element of \( K(\Lambda)/S \). Note that \( K(\Lambda)/S \) is of odd order.

We claim that \( K(H)/S \) splits over \( K(\Lambda)/S \). Let \( r \) be a prime and \( R/S \) be the \( r \)-Sylow subgroup of \( K(H)/S \), \( K(H) \supseteq K(\Lambda) \), and \( R_0/S = R/S \cap K(\Lambda)/S \). If \( r = 2 \) then \( R_0/S = 1 \). Thus let \( r \) be odd, \( R_0/S = \langle cS \rangle \) where \( cS \) is of order \( r^n \), and \( R/R_0 = \langle z_1R_0, \ldots, z_nR_0 \rangle \), \( z_iR_0 \) being a basis for the abelian group \( R/R_0 \). Let \( z = z_i \), and assume \( zR_0 \) is of order \( r^u \) with \( z^{r^u} \equiv c^r \) (mod \( S \)). If \( u = v \), then, letting \( w = z^{-1}c^{r^{-u}} \), we get \( w^{r^u} \equiv 1 \) (mod \( S \)). Assume \( v < s \) and \( u > v \). Since \([R, y] \subseteq G \cap R \subseteq K(\Lambda) \cap R = R_0 \), it follows that \( y^{-1}zy \equiv zc^\lambda \) (mod \( S \)). Thus

\[ y^{-1}z^{r^u}y \equiv z^{r^u}c^{r^u} \equiv c^{-r^u} \quad \text{(mod } S \text{)}, \]

implying \( \lambda r^u \equiv -2r^v \) (mod \( r^s \)), a contradiction. Thus \( K(H)/S \) splits over \( K(\Lambda)/S \).

The above also implies that \( c, z_1, \ldots, z_n, S \), with the \( z_i \)'s chosen such that \( z_iS \) is of order \( r^{u_i} \), form a basis for \( R/S \) with \( R_0/S = \langle cS \rangle \). (Note that if \( r = 2 \) then \( R_0/S = 1 \), and thus the above is true.) Let \( y^{-1}z_iy \equiv z_i\lambda \) (mod \( S \)) and \( \mu_i \) be a solution of \( \lambda_i + 2\mu_i \equiv 0 \) (mod \( r^s \)). If \( R_0/S = 1 \) (as in the case of \( r = 2 \)) let \( \lambda_i = \mu_i = 0 \). For each \( r \)-Sylow subgroup \( R/S \) of \( K(H)/S \), we fix such a set of \( z_i \)'s.

Our aim now is to construct the character \( \theta \) of the theorem. Let \( T \) be a (complex) linear representation of \( K(\Lambda) \), \( \overline{T} = \sigma, \) ker \( T \cap G' = H, \) and \( T(k) = 1 \) for every \( 2 \)-element \( k \) of \( K(\Lambda) \). Then ker \( T = \ker \sigma = S \). Extend \( T \) as \( T' \) to all elements of \( RK(\Lambda) \) for every \( r \)-Sylow subgroup \( RK(\Lambda)/K(\Lambda) \) of \( K(H)/K(\Lambda) \) and then to all elements of \( K(H) \) as follows: If \( R_0/S = 1 \) (as in the case of \( r = 2 \)) then let \( T'(z_i) = 1 \) for all \( i \). If \( R_0/S \neq \langle cS \rangle \) (notations as above) let \( T'(z_i) = T(c) \). Now if \( k \in K(H) \) then \( k = a[I'] \), \( I' \) running over all primers \( r \) dividing \( |K(H)/S| \) and \( a \in K(\Lambda) \). Define \( T'(k) = T(a)(I' \cdot T'(z_i)^{\alpha_i}) \). It is easy to show that \( T' \) is a linear representation of \( K(H) \). Now \( T' \in R(K(\Lambda), S, K(\Lambda)) \), as in the notations of \([1, p. 100]\), and hence \( T'^G \) is irreducible. Since \( \overline{T'}(K(\Lambda)) = \sigma \) we have \( T'^G \in B(\sigma, H) \). Let \( \theta \) be the character of \( T'^G \). Then \( \theta \) is \( 2 \)-rational and since ker \( \theta \) contains the defect group of \( B(\sigma, H) \), \( \theta \) is of height zero in \( B(\sigma, H) \). It remains to show that \( Q(B(\sigma, H)) = Q(\theta) \).

Now \( I(T') = K(H), I(T') \) the inertia group of \( T' \). We claim that \( I(\sigma) = K(H) \). Let \( x \in I(\sigma) \). Then for any \( r \), \( T(x^{-1}cx) = T(c) \). (Here \( c, z_i \) depend on \( r \) as in the notations above.) If \( R_0/S = 1 \), then \( x^{-1}z = z \) (mod \( S \)) for any \( z = z_i \in R \). Assume for some (odd) \( r \), \( R_0/S \neq \langle cS \rangle \) -1, and for some generator \( zS \) of \( R/S \), \( x^{-1}z \equiv zc^\delta \) (mod \( S \)). Since \( x^{-1}z \equiv z \) (mod \( S \)), \( x \not\equiv y \). Thus we may assume \( x = y \equiv x \) (mod \( S \)), \( a \in K(\Lambda) \). It follows that \( (xy)^{-1}z(\lambda) \equiv (yx)^{-1}z(\lambda) \) (mod \( S \)). After a short computation we have \( zc^\lambda \equiv zc^\delta \equiv zc^\lambda \) (mod \( S \)), \( \lambda \) defined as above. Thus \( 2\delta \equiv 0 \) (mod \( r^s \)), and since \( r \neq 2, \delta \equiv 0 \) (mod \( r^s \)) or \( x \in K(\Lambda) \). Thus \( I(\sigma) = K(H) = I(T') \).

Let \( \xi \) be a primitive \(|G| \)th root of unity. Then \( Q(\xi) \subseteq Q(\xi) \) for all irreducible characters \( \chi \) in \( B(\sigma, H) \). Let \( \tau \in \hat{Q}(Q(\xi)/Q) \) and assume \( B(\sigma, H)^\tau = B(\sigma, H) \). This
implies both $T'^G$ and $(T'^G)'$ are in $B(\sigma, H)$ and thus $\sigma = \tau = \bar{T}' = (\bar{T}')_K \Lambda$ and $\bar{T}' = \bar{T}'_K = (\bar{T}')_K \Lambda$, are $G$-conjugate or $T$ and $T'$ are $G$-conjugate. Assume $T' = T^g$ for some $g \in G$. Since $\ker T = \ker T^g = S$, it follows that $g \in N(S)$. Now, for the prime $r$, $r \neq 2$, the group $\langle y, g \rangle$ induces a (cyclic) automorphism subgroup on $R_0/S = \langle cS \rangle$. That is, there is an element $h_r \in N(S)$ such that $\langle h_r \rangle$ induces the same automorphism group on $R_0/S$ as $\langle y, g \rangle$. If $h_r$ induces an automorphism of order $n$ on $R_0/S$ then $n$ is even and $h_r^{-n/2}$ inverts every element of $R_0/S$. Assume $h_r^{-1}ch_r \equiv c^a (mod S)$. Then $r \mid a - 1$, $g^{-1}cg \equiv c^{a'} (mod S)$ and $a^{n/2} \equiv -1 (mod r^s)$. Assume $h_r^{-1}zh_r \equiv zc^a (mod S)$, $g^{-1}zg \equiv zc^b (mod S)$, where $zS$ is a generator of $R/S$ as defined above. Since

\[
(yh_r^{n/2})^{-1}z(yh_r^{n/2}y) \equiv (h_r^{n/2}y)^{-1}z(h_r^{n/2}z) \quad (mod S),
\]

\[
(gh_r^{n/2})^{-1}z(gh_r^{n/2}) \equiv (h_r^{n/2}g)^{-1}z(h_r^{n/2}g) \quad (mod S),
\]

and

\[
h_r^{-n/2}zh_r^e \equiv c^{a^e} \equiv (\sigma^{-1} + \cdots + 1) \quad (mod S) \quad \text{for any } e,
\]

it follows that

\[
\lambda \equiv \nu(a^{n/2} - 1 + \cdots + 1) \quad (mod r^s) \quad \text{and} \quad \delta \equiv \nu(a^{r-1} + \cdots + 1) \quad (mod r^s).
\]

That is, $\langle y, g \rangle$ induces a cyclic automorphism group on $R/S$. Recalling that $T'(z) = T(c)^\alpha$, $\lambda + 2\mu \equiv 0 (mod r^s)$, and $T' = T^g$, we have

\[
T'(z) = [T(c)^\alpha] = T(g^{-1}cg)^\mu = T(c)^{\alpha \mu}.
\]

Also

\[
T'(g^{-1}zg) = T'(zc^b) = T'(z)T(c)^\delta = T(c)^{\mu + \delta}.
\]

We need to prove $\mu \alpha' \equiv \mu + \delta (mod r^s)$ or $\mu(\alpha' - 1) \equiv \delta (mod r^s)$. Since $a^{n/2} \equiv -1 (mod r^s)$, we have

\[
2\nu(\alpha' - 1) \equiv -\nu(\alpha' - 1)(a^{n/2} - 1) \quad (mod r^s).
\]

Since $r \mid \alpha - 1$, we have

\[
2\nu(\alpha'-1 + \cdots + 1) \equiv -\nu(\alpha' - 1)(a^{n/2} + \cdots + 1) \quad (mod r^s).
\]

Thus

\[
2\delta \equiv -\lambda(\alpha' - 1) \quad (mod r^s) \quad \text{or} \quad \delta \equiv \mu(\alpha' - 1) \quad (mod r^s)
\]

as desired. Thus $T'^G = T^g$ or $T'^G$ and $T'$ are $G$-conjugate. This implies $T'^G$ and $(T'^G)^G$ are equivalent or $\theta^* = \theta$. The proof is complete.

The above implies

**Corollary.** Let $B$ be a 2-block of a finite metabelian group and assume $B$ contains all the $Q$-conjugates of some irreducible character. Then $B$ contains an irreducible rational character of height zero.
REFERENCES


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