A REPRESENTATION THEOREM FOR SEMILATTICES

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Abstract. We prove that every semilattice \((L, \wedge)\) admits an embedding \(Q\) into the set \(R(X)\) of all partial orders on some set \(X\) such that for all \(a, b \in L\), \(Q(a \wedge b) = Q(a) \cap Q(b)\) and if \(a \vee b\) exists then also \(Q(a \vee b) = Q(b) \circ Q(a)\).

Denote by \(R(X)\) the set of all partial orders on \(X\). The set \(R(X)\) is closed under the operation of intersection of binary relations. Therefore we can consider \(R(X)\) as a semilattice \((R(X), \cap)\), which we call the semilattice of partial orders on \(X\). The relation product \(S \circ H\) of two partial orders \(H, S \in R(X)\) need not belong to \(R(X)\). But if it does then \(S \circ H\) is the least upper bound of \(\{H, S\}\) in \(R(X)\) under the set-theoretical inclusion.

By a representation of a semilattice \((L, \wedge)\) we mean an embedding \(F: L \rightarrow R(X)\) for some set \(X\) such that \(F(a \wedge b) = F(a) \cap F(b)\) for all \(a, b \in L\). The set \(X\) is called the domain of the representation \(F\).

The main result of this note is the following theorem.

Theorem. Every semilattice \((L, \wedge)\) admits a representation \(Q\) such that for all \(a, b \in L\), if \(a \vee b\) exists then \(Q(a \vee b) = Q(b) \circ Q(a)\).

The proof of this theorem uses some techniques of B. Jónsson [2]. Let \((L, \wedge)\) be an arbitrary semilattice.

Lemma 1. \((L, \wedge)\) has a representation.

Consider the mapping \(F: L \rightarrow R(L)\) defined by

\[
F(a) = \{(x, y) \in L \times L | x \leq y \text{ and } x, y \leq a\} \cup \Delta_L,
\]

where \(\Delta_L\) denotes the identity relation on \(L\). It is easy to see that \(F\) is a representation of the semilattice \((L, \wedge)\).

We shall say that the representation \(P\) with domain \(Y\) is an extension of a representation \(F\) with domain \(X\) if \(X \subset Y\) and \(F(a) = P(a) \cap X \times X\) for all \(a \in L\).

Lemma 2. If \(F\) is a representation of \((L, \wedge)\), \(a \vee b\) exists, and \((p, q) \in F(a \vee b)\) where \(p \neq q\), then there exists an extension \(P\) of \(F\) such that \((p, q) \in P(b) \circ P(a)\).

Let \(Y = X \cup \{r\}\) where \(X\) is the domain of representation \(F\) and \(r\) does not belong to \(X\). Let \(x, y\) denote arbitrary elements of \(X\), and define \(P\) by putting

\[
(x, y) \in P(c) \iff (x, y) \in F(c),
\]

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(2) \((x, r) \in P(c) \Rightarrow (x, p) \in F(c)\) and \(a \leq c\),
(3) \((r, y) \in P(c) \Rightarrow (q, y) \in F(c)\) and \(b \leq c\),
(4) \((r, r) \in P(c)\).

It is easy to see that \(P(c) \in R(Y)\) for all \(c \in L\). From (1) we see that \(P(c) = F(c) \cap X \times X\), and the fact that \(F\) is one-to-one implies the same for \(P\). Using (1)–(4) it is easily seen that \(P(c \land d) = P(c) \cap P(d)\) for all \(c, d \in L\). Therefore \(P\) is a representation. Since \((p, r) \in P(a)\) and \((r, q) \in P(b)\), we have \((p, q) \in P(b) \circ P(a)\), which completes the proof of Lemma 2.

Suppose that \(\alpha\) is an inaccessible cardinal such that \(|L| < \alpha\) where \(|L|\) is the power of the set \(L\). Denote by \(\Sigma\) the set of all representations \(F: L \to R(X)\) such that \(|X| < \alpha\). By Lemma 1, \(\Sigma \neq \emptyset\). Suppose that \(\{F_i|i \in I\}\) is a family of representations belonging to \(\Sigma\) such that the family \(\{X_i|i \in I\}\) of the corresponding domains is a chain under set-theoretical inclusion and \(F_i\) is an extension of \(F_j\) if \(X_i \subset X_j\). Then it is easy to see that the mapping \(F: L \to R(X)\) where \(X = \bigcup \{X_i|i \in I\}\) and \(F(c) = \bigcup \{F(c_i)|i \in I\}\) for all \(c \in L\), is a representation belonging to \(\Sigma\). Hence by the Zorn Lemma there exists a representation \(Q\) which is maximal in the sense that if \(P\) is an extension of \(Q\) then \(P\) equals \(Q\).

Let \(a, b \in L\) and suppose that \(a \lor b\) exists. First we show that \(Q(a \lor b) \subset Q(b) \circ Q(a)\). Suppose the contrary, i.e. that there exist \(x, y\) such that \((x, y) \in Q(a \lor b)\) and \((x, y) \notin Q(b) \circ Q(a)\). Then, by Lemma 2, there exists an extension \(P\) of \(Q\) such that \((x, y) \in P(b) \circ P(a)\), but this contradicts the maximality of \(Q\). Thus \(Q(a \lor b) \subset Q(b) \circ Q(a)\). Since \(a \leq a \lor b\) and \(b \leq a \lor b\), we have \(Q(a) \subset Q(a \lor b)\) and \(Q(b) \subset Q(a \lor b)\). Therefore \(Q(b) \circ Q(a) \subset Q(a \lor b) \circ Q(a \lor b) \subset Q(a \lor b)\). Thus \((a \lor b) = Q(b) \circ Q(a)\), and this completes the proof of the theorem.

From this theorem the following result of [1,3] is obtained immediately.

**Corollary.** Every lattice \((L, \land, \lor)\) is isomorphic to an algebra of binary relations \((\mathcal{P}, \cap, \circ)\), where all elements of \(\mathcal{P}\) are partial orders.

Note that the construction used in the proof of our theorem is simpler than the one in [1,3].

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**REFERENCES**