PROBABILITY MEASURE REPRESENTATION OF NORMS ASSOCIATED WITH THE NOTION OF ENTROPY

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Abstract. One of the applications of Banach spaces introduced by B. Korenblum [1, 2] is a new convergence test [2] for Fourier series including both Dirichlet-Jordan and the Dini-Lipschitz tests [3]. The norms of the spaces are given in terms of \( \kappa \)-entropy where \( \kappa(s) > 0, 0 < s \leq 1 \), is a nondecreasing concave function such that \( \kappa(1) = 1 \). The \( \kappa \)-norms fill the gap between the uniform and the variation norms. The original proof of the general properties of \( \kappa \)-norms uses both combinatorial and approximation arguments which are rather complicated. We give a simple proof introducing a probabilistic representation of the norms so that the \( \kappa \)-norm of a real function \( f \) on \( T = \mathbb{R}/2\pi\mathbb{Z} \) is the expectation of the mean oscillation of \( f \) on a subinterval of \( T \), chosen in a suitable random process.

The \( \kappa \)-entropy. We start with the definition of the \( \kappa \)-norm introduced by B. Korenblum [1, 2]. Let \( T = \mathbb{R}/2\pi\mathbb{Z} \), and let \( |E| = \int_E dx \) denote the normalized Lebesgue measure of a Borel subset \( E \) of \( T \). The distance between points \( x, y \in T \) is \( d(x, y) = \min\{|x - y + 2\pi n|; n \in \mathbb{Z}\}/2\pi \). \( L^\infty(T) \) is the space of complex valued essentially bounded function on \( T \), and \( RL^\infty(T) \) denotes the space of real-valued function in \( L^\infty(T) \). The graph of a function \( f \in RL^\infty(T) \) is the set

\[
\Gamma(f) = \{(t, y) \in T \times R; \lim_{\delta \to 0} \left( \inf_{\delta} \{f(t); d(t, \tau) < \delta\} \right) \leq y \leq \lim_{\delta \to 0} \left( \sup_{\delta} \{f(\tau); d(t, \tau) < \delta\} \right) \}.
\]

One shows that \( \Gamma(f) \) is always a closed subset of \( T \times R \) which is connected on any subinterval of \( T \).

Definition 1. Let \( \kappa(s) > 0, 0 < s \leq 1 \), be a positive nondecreasing concave function such that \( \kappa(1) = 1 \). The \( \kappa \)-entropy of a finite subset \( E \) of \( T \) \((E \neq \emptyset)\) is \( \kappa(E) = \sum_{i=1}^{n} \kappa(|I_i|) \) where \( \{I_i\}_{i=1}^{n} \) are the complementary intervals of \( E \). For an infinite closed subset \( E \) of \( T \) \((E = \emptyset)\), we set \( \kappa(E) = \sup\{\kappa(F); F \subset E, F \text{-finite}\} \). We also put \( \kappa(\emptyset) = 0 \).

Definition 2. For any function \( f \in RL^\infty(T) \) we set

\[
\|f\|_\kappa = \int_{\infty}^{\infty} \kappa(\Gamma_y(f)) dy
\]

where \( \Gamma_y(f) = \{(t; (t, y) \in \Gamma(f))\} \) is the \( y \)-level set of the graph of \( f \).

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Notice that for a continuous real function \( f \) on \( T \),
\[
\| f \|_\kappa = \| f \|_C = \max\{ f(t); \ t \in T \} - \min\{ f(t); \ t \in T \}
\]
if \( \kappa(s) = s \),
and
\[
\| f \|_\kappa = \| f \|_\nu = \sup \left\{ \sum_{i=1}^n |f(t_i) - f(t_{i-1})|; \ t_0 = 0 < t_1 < \ldots < t_n = 2\pi \right\}
\]
if \( \kappa(s) = 1 \).

For an arbitrary function \( \kappa \) as in Definition 1 we have \( s \leq \kappa(s) \leq 1 \) for \( 0 < s \leq 1 \)
and, consequently, \( \| \cdot \|_C \leq \| \cdot \|_\kappa \leq \| \cdot \|_\nu \). If \( \kappa(0^+) = \lim_{s \to 0}(\kappa(s) = \alpha > 0 \), then \( \kappa = \alpha + \beta\kappa_0 \) where \( \alpha + \beta = 1 \) and \( \kappa_0(0^+) = 0 \). In this case \( \| \cdot \|_\kappa = \alpha\| \cdot \|_\nu + \beta\| \cdot \|_\kappa_0 \).

If \( \kappa(0^+) = 0 \) and \( \lim_{s \to 0}(\kappa(s)/s) = A < \infty \), then \( \| \cdot \|_\kappa \leq A\| \cdot \|_C \). i.e. \( \| \cdot \|_\kappa \) is equivalent to the norm \( \| \cdot \|_C \). We, therefore, assume in what follows that \( \kappa(0^+) = 0 \) and \( \lim_{s \to 0}(\kappa(s)/s) = \infty \). In this case \( \| f \|_\kappa = \int_{(T \times R)}\kappa'(f(t), x)) dx \, dy \) where \( \kappa(s) = \int_0^s\kappa'(t) \, dt \). It is by no means simple to show that \( \| \cdot \|_\kappa \) is, in fact, a norm using just Definition 2 (the triangle inequality is hard to establish). We give another representation of \( \kappa \)-norms so that many properties of these norms follow naturally.

**Probability measure representation of \( \kappa \)-norms.** For closed subsets \( A, B \) of \( T \times R \) we set \( \lambda A = \{(t, \lambda y); (t, y) \in A\}, \) \( A + B = \{(t, y + y'); (t, y) \in A, (t, y') \in B\} \)
and \( AB = \{(t, yy'); (t, y) \in A, (t, y') \in B\} \). For \( t \in T \) and \( 0 \leq s \leq \frac{1}{2} \) we define the oscillation of \( A \) on the interval \( \{t; d(t, t) \leq s\} \) by the formula
\[
\Omega_A(t, s) = \max\{ y; (t, y) \in A, d(t, t) \leq s\} - \min\{ y; (t, y) \in A, d(t, t) \leq s\}.
\]
We also set \( \| A \|_\infty = \max\{\| y \|; (t, y) \in A\} \). One easily verifies \( \Omega_{\lambda A} = |\lambda|\Omega_A \) for any \( \lambda \in R \), \( \Omega_{A + B} \leq \Omega_A + \Omega_B \), \( \Omega_A \leq |A\|_\infty\Omega_B + \| B \|_\infty\Omega_A \). For a function \( f \in RL^\infty(T) \) we put \( \Omega_f = \Omega_{f(t)} \). The following Lemma follows easily from the properties of \( \Omega \) listed above.

**Lemma.** Let \( f, g \in RL^\infty(T) \). We then have

(i) \( \Omega_{\lambda f} = |\lambda|\Omega_f, \lambda \in R \);
(ii) \( \Omega_{f + g} \leq \Omega_f + \Omega_g \);
(iii) \( \Omega_{fg} \leq \| f \|_\infty\Omega _g + \| g \|_\infty\Omega _f \);
(iv) if \( f \) is continuously differentiable, then \( \Omega_f(t, s) \leq 2s\| f' \|_\infty \).

**Definition 3.** Let \( \mu \) be a probability measure on the unit interval \([0, 1]\). The \( \mu \)-norm of a function \( f \in RL^\infty(T) \) is
\[
\| f \|_\mu = \int_{T \times [0, 1]} \Omega_f(t, s/2) \, d\mu(s) \, dt,
\]
where the value of \( \Omega_f(t, s/2)/s \) at 0 is \( \lim_{s \to 0}(\Omega_f(t, s/2)/s) \). Notice that if \( \mu \) is concentrated at 1, then \( \| \cdot \|_\mu = \| \cdot \|_C \). If \( \mu \) is concentrated at 0, and \( f \) is an absolutely continuous function on \( T \), then \( \| f \|_\mu = \| f \|_\nu \).
Theorem 1. Let $\kappa(s), 0 < s \leq 1$, be a positive nondecreasing concave function such that $\kappa(0^+) = 0$ and $\lim_{s \to 0} (\kappa(s)/s) = \infty$. There is a unique probability measure $\mu_\kappa$ on $[0, 1]$ such that

$$\|f\|_\kappa = \|f\|_{\mu_\kappa} = \int_{T \times [0, 1]} \left( \frac{\Omega_f}{s} \right) \left( t, \frac{s}{2} \right) dt d\mu_\kappa(s).$$

The map $\kappa \to \mu_\kappa$ gives a one-to-one correspondence between the sets $K_0 = \{ \kappa; \kappa$ described above$\}$ and $P_0 = \{ \mu; \mu$ a probability measure on $[0, 1]$ such that $\mu(\{0\}) = 0$, $\int_0^1 \mu(\tau)/\tau^2 d\tau = \infty \}$. 

Remark. If $\mu$ is a probability measure on $[0, 1]$ such that $\mu(\{0\}) = 0$ then the condition $\int_0^1 \mu(\tau)/\tau^2 d\tau = \infty$ is equivalent to the condition

$$\lim_{s \to 0} \frac{1}{s} \int_s^1 \frac{d\mu(\tau)}{\tau} d\tau = \infty.$$

For the proof of the theorem we will use this condition rather than the condition $\int_0^1 \mu(\tau)/\tau^2 d\tau = \infty$.

Proof. Let $\kappa \in K_0$, $\kappa'(s), 0 \leq s \leq 1$, is a nonnegative nonincreasing left-continuous function such that $\kappa'(0) = \infty$ and $\kappa(s) = \int_0^s \kappa'(t) dt$. We define $\alpha_\kappa(z) = \max\{s; \kappa'(s) \geq z\} = |\{s; \kappa'(s) \geq z\}|$. The function $\alpha_\kappa$ has the properties

(*) $\alpha_\kappa(z) > 0$ for all $z \geq 0$,

(**) $\kappa'(s) \geq z$ if and only if $s \leq \alpha_\kappa(z)$ for all $s \in [0, 1]$, $z \geq 0$,

(***) $\kappa'(s) = \max\{z; \kappa'(s) \geq z\} = |\{z; \alpha_\kappa(z) \geq s\}|$.

The measure $\mu_\kappa$ is defined by the formula $d\mu_\kappa(s) = sda^{-1}_\kappa(s)$ where $\int_E d\alpha^{-1}_\kappa(s) = |\alpha^{-1}_\kappa(E)|$ for any Borel subset $E$ of $[0, 1]$. For any $0 < s \leq 1$,

$$\mu_\kappa([0, s]) = \int_0^s t d\alpha^{-1}_\kappa(t) = \int_0^\infty \chi_{[0, s]}(\alpha_\kappa(z)) \alpha_\kappa(z) dz;$$

hence,

$$\mu_\kappa([0, 1]) = \int_0^\infty \alpha_\kappa(z) dz = \int_0^\infty |\{s; \kappa'(s) \geq \}| dz = \int_0^\infty \kappa'(s) ds = \kappa(1) = 1.$$

By the dominated convergence theorem $\mu_\kappa(\{0\}) = \lim_{s \to 0} \mu_\kappa([0, s]) = 0$. We show that for any function $f \in RL^\infty(T), \|f\|_\kappa = \|f\|_{\mu_\kappa},$

$$\|f\|_\kappa = \int_{T \times R} \kappa'(2d(\Gamma_y(f), x)) dx dy = \int_{T \times R^+} |\{y; \kappa'(2d(\Gamma_y(f), x)) \geq z\}| dx dz$$

$$= \int_{T \times R^+} |\{y; d(\Gamma_y(f), x) \leq \alpha_\kappa(z)/2\}| dx dz$$

because of the property (**). Notice that $|\{y; d(\Gamma_y(f), x) \leq s\}| = \Omega_f(x, s).$ We then get

$$\|f\|_\kappa = \int_{T \times R^+} \Omega_f(x, \alpha_\kappa(z)/2) dx dz = \int_{T \times [0, 1]} \frac{1}{s} \Omega_f(x, s/2) d\mu_\kappa(s) dx,$$
since \( \mu_a((0)) = 0 \). We now prove the second part of the theorem. Let \( \kappa \in K_0 \) and \( \mu = \mu_a \). We know that \( \mu \) is a probability measure and \( \mu((0)) = 0 \). We set

\[
\kappa_\mu(s) = \int_0^s \int_t^1 \frac{1}{\tau} d\mu(\tau) \, dt.
\]

By property (*** and the definition of \( \mu = \mu_a \),

\[
\kappa_\mu(s) = \int_0^s \int_t^1 d\kappa_a^{-1}(\tau) \, d\tau \, dt = \int_0^s |\kappa_a^{-1}([t, 1])| \, dt = \int_0^s \kappa'(t) \, dt = \kappa(s).
\]

This shows that \( \mu = \mu_a \in P_0 \) and \( \kappa_\mu = \kappa_\mu = \kappa \).

Assume now that \( \mu \in P_0 \) and let

\[
\kappa(s) = \kappa_\mu(s) = \int_0^s \int_t^1 \frac{1}{\tau} d\mu(\tau) \, dt.
\]

One can see that \( \kappa \in K_0 \). We next have \( \kappa'(s) = \int_t^1 \frac{1}{\tau} d\mu(\tau) / \tau \), and by property (***)

\[
\kappa'(s) = \int_0^s d\kappa_a^{-1}(\tau) = \int_0^s \frac{d\mu_a(\tau)}{\tau}. \quad \text{Consequently, } \mu = \mu_a \text{ since also } \mu((0)) = \mu_a((0)) = 0. \]

This shows that the map \( \mu \to \kappa_\mu \) is the inverse of the map \( \kappa \to \mu_a \).

**Remarks.** If \( \kappa \in K_0 \) and \( \kappa' \) is continuous strictly decreasing and \( \kappa'(1) = 0 \), then \( \alpha_a \) is the inverse function of \( \kappa' \). If \( \kappa \in K_0 \) is twice differentiable and \( \kappa' \) is strictly decreasing then \( d\mu_a(s) = \kappa'(1) d\delta_1(s) - s\kappa''(s) \, ds \) where

\[
\delta_1(E) = \begin{cases} 1 & \text{if } 1 \in E, \\ 0 & \text{if } 1 \notin E. \end{cases}
\]

For example, if

1. \( \kappa(s) = s^\alpha, \quad 0 < \alpha < 1 \) (\( \kappa(E) \) is the Lipschitz entropy), then \( d\mu_a(s) = \alpha d\delta_1(s) + (1 - \alpha)2^\alpha s^{\alpha - 1} \, ds \);
2. \( \kappa(s) = s(|\log s| + 1) \) (\( \kappa(E) \) is the Shannon entropy), then \( d\mu_a(s) = ds \);
3. \( \kappa(s) = (1 + \frac{1}{2} |\log s|)^{-1} \) (\( \kappa(E) \) is the Dini entropy), then \( d\mu_a(s) = \frac{1}{2} d\delta_1(s) + \frac{1}{2} \frac{|\log s|}{s(1 + |\log s|/2)^3} \, ds \).

**Corollary.** \( || \cdot ||_\kappa \) is homogeneous and satisfies the triangle inequality for any \( \kappa \in K_0 \). Moreover, if \( f \) is a real continuously differentiable function on \( T \), then

\[
|| f ||_\kappa \leq || f ||_\infty \kappa(|| f ||_C / || f' ||_\infty).
\]

**Proof.** The first part of the Corollary follows directly from the probability representation of the \( \kappa \)-norms and from the lemma. For the proof of the second part we let \( f \) be a nonconstant differentiable function on \( T \), so that \( || f' ||_\infty < \infty \). We denote \( A = || f ||_C \) and \( B = || f' ||_\infty \) \((B \neq 0)\). Using the probability representation of the \( \kappa \)-norm we obtain

\[
|| f ||_\kappa = \int_{T \times [0,1]} \frac{\Omega_f(t, s/2)}{s} \, d\mu(s) \, dt = B \int_{T \times [0,1]} \frac{\Omega_{f'/B}(t, s/2)}{s} \, d\mu(s) \, dt
\]

\[
= B \int_{T \times 0}^\infty \Omega_{f'/B}(t, \alpha_a(y)/2) \, dy \, dt
\]
where \( \alpha_k \) is the function defined in the proof of Theorem 1. We notice that \( \Omega_{f/B}(t, s/2) \) is nondecreasing and right-continuous in \( s \). We define \( \beta(y) = \min\{s; \Omega_{f/B}(t, s/2) \geq y\} \) for \( 0 \leq y \leq A/B \). Clearly \( \Omega_{f/B}(t, s/2) \geq y \) if and only if \( s \geq \beta(y) \). Moreover, \( \beta(y) \geq y \) for all \( 0 \leq y \leq A/B \). Therefore

\[
\|f\|_x = B \int_{T_0}^{A/B} \left\{ \int_0^{\beta(y)} \alpha_k(z) \, dz \right\} \, dy \, dx = B \int_{T_0}^{A/B} k'(\beta(y)) \, dy \, dt
\]

\[
\leq B \int_{T_0}^{A/B} k'(y) \, dy \, dt = Bk(A/B).
\]

In what follows \( \kappa \) is again a function from the set \( K_0 \). We define the following linear spaces with the norm \( \| \cdot \|_x + \| \cdot \|_k \): \( RL^\infty_k(T) = \{ f \in RL^\infty(T); \| f \|_k < \infty \} \)
and \( RC_k = \{ f; f \) is continuous on \( T, \| f \|_k < \infty \}. \) The analogous spaces \( L^\infty_k \) and \( C_k \)
of complex valued functions are defined as the complexification of the real spaces \( RL^\infty_k \) and \( RC_k \), respectively (see, for example, [4]). We will use the probability measure representation of \( \kappa \)-norms to prove some general properties of \( L^\infty_k \) and \( C_k \).

**Theorem 2.** (a) The spaces \( L^\infty_k \) and \( C_k \) are Banach algebras with the usual multiplication of functions.

(b) \( C_k \) is the largest translation invariant subspace of \( L^\infty_k \) on which the shift operator \( T_x f(t) = f(t - x) \) has the property \( ||T_x f - f||_k \to 0\), if \( x \to 0 \).

(c) The polynomials are dense in \( C_k \). In particular, \( C_k \) is separable.

**Proof.** (a) It is enough to show that \( (RL^\infty_k, \| \cdot \|_x + \| \cdot \|_k \) \) is a Banach algebra. Submultiplicativity of the norm \( \| \cdot \|_x + \| \cdot \|_k \) follows directly from Theorem 1 and (iii) of the Lemma. To show completeness of \( RL^\infty_k(T) \) take a sequence \( \{f_n\}_{n=1}^\infty \) in \( RL^\infty_k \) which is Cauchy in the norm \( \| \cdot \|_x + \| \cdot \|_k \). If \( f \) is the uniform limit of \( f_n \) then \( \Omega_{f_n-f}(t, s) \to \Omega_{f-f}(t, s) \) as \( m \to \infty \) uniformly in \( (t, s) \). By Fatou’s lemma,

\[
||f - f_n||_k = \int_{T \times [0,1]} \Omega_{f-f_n}(t, s/2) \frac{d\mu_k(s)}{s} \, dt \leq \lim_{m \to \infty} \int_{T \times [0,1]} \Omega_{f_m-f_n}(t, s/2) \frac{d\mu_k(s)}{s} \, dt
\]

which shows that \( ||f - f_n||_k \to 0 \) as \( n \to \infty \).

(b) Let \( f \in RC_k \). We must prove that \( \lim_{x \to 0} ||T_x f - f||_k = 0 \). Pick \( 0 < \delta < 1 \).

\[
\lim_{x \to 0} ||T_x f - f||_k = \lim_{x \to 0} \left( \int_{T_0}^{\delta} \Omega_{T_x f-f}(t, s/2) \frac{d\mu_k(s)}{s} \, dt \right)
\]

\[
+ \lim_{x \to 0} \left( \int_{T_0}^{\delta} \Omega_{T_x f-f}(t, s/2) \frac{d\mu_k(s)}{s} \right) \, dt
\]

\[
\leq \lim_{x \to 0} \int_{T_0}^{\delta} (\Omega_{T_x f} + \Omega_f) \frac{d\mu_k(s)}{s} \, dt + \lim_{x \to 0} \left( \int_{T_0}^{\delta} \Omega_{T_x f-f} \frac{d\mu_k(s)}{s} \, dt \right)
\]

\[
= 2 \int_{T_0}^{\delta} \frac{d\mu_k(s)}{s} \, dt.
\]
By the dominated convergence theorem,

\[
\lim_{x \to 0} \|T_x f - f\| \leq 2 \lim_{\delta \to 0} \int_{-\delta}^{\delta} \frac{\Omega_f(s)}{s} dt = 0.
\]

(c) By part (b) we have that \( C_{\Omega} \) is a homogeneous Banach space on \( T \) [5, Definition 2.10] and, consequently, polynomials are dense in it [5, Theorem 2.12].

**Remark.** We will give the following probabilistic interpretation of the \( \mu \)-norms. We will pick an interval \( I = \{\tau; d(t, \tau) < s/2\} \) in \( T \); the center \( t \) of \( I \) is chosen with the probability evenly distributed along \( T \) and the length \( |I| = s \) is chosen with the probability of the distribution \( \mu(s) = \int_0^s d\mu(\tau) \). The \( \mu \)-norm of a function \( f \) on \( T \) is simply the expectation of the random variable \( X(I) = \text{mean oscillation of } f \text{ on } I = \Omega_f(t, s/2)/s \) in this process.

**References**

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