REFLEXIVITY OF A BANACH SPACE
WITH A UNIFORMLY NORMAL STRUCTURE

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Abstract. In this note we prove that any Banach space with a uniformly normal structure is reflexive.

1. Introduction. In [4] Gillespie and Williams gave the concept of uniformly normal structure of Banach spaces. They showed that any nonexpansive self-map of a closed convex bounded subset of a Banach space with a uniformly normal structure has a fixed point, and, in [5], obtained the same result for the Kannan-type maps.

In this note, we show that any Banach space with a uniformly normal structure is reflexive, and, consequently, the main results of Gillespie and Williams are actually contained in those of Kirk [9], Godhe [6] and Kannan [8], respectively.

2. Main result. A Banach space $X$ is said to have a uniformly normal structure if there exists a number $h$, $0 < h < 1$, such that if $C$ is a closed convex bounded subset of $X$, then there exists $x$ in $C$ such that $\sup\{\|x - y\|; y \in C\} \leq h \delta(C)$, where $\delta(C)$ denotes the diameter of the set $C$.

To prove our theorem, we adopt the idea of Huff [7].

Theorem. Any Banach space with a uniformly normal structure is reflexive.

Proof. We use a theorem of Eberlein and Smulian [2, p. 51]. Let $\{K_n\}$ be a decreasing sequence of nonvoid closed convex bounded subsets of a given Banach space with a uniformly normal structure. We need to show that $\bigcap K_n \neq \emptyset$. For each $n$, choose $x_n \in K_n$. Call a sequence $\{y_n\}$ a $c$-subsequence of $\{x_n\}$ provided there exists a sequence of integers $1 = p_1 \leq q_1 < p_2 \leq q_2 < \cdots$ and coefficients $\alpha_i \geq 0$ such that, for each $n$,

$$\sum_{i=p_n}^{q_n} \alpha_i = 1, \quad y_n = \sum_{i=p_n}^{q_n} \alpha_i x_i.$$ 

Then for each $\varepsilon > 0$, there exists a $c$-subsequence $\{y_{n_m}\}$ of $\{x_n\}$ with $\|y_{n_m} - y_{m}\| < \varepsilon$ for each $n, m$. Suppose this is not true for some $\varepsilon > 0$.

Let $L_m = \{x_n\}_{n=m}^{\infty}$. Let $\text{co}(L_m)$ and $\overline{\text{co}}(L_m)$ denote the convex hull and the closed convex hull of $L_m$, respectively. Then there exists $h$, $0 < h < 1$, and $y'_1 \in \overline{\text{co}}(L_1)$ such

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that \( \sup(\|y'_1 - y\|; y \in \overline{\text{co}}(L_1)) \leq h\delta(L_1) \). Let \( 0 < h < h_1 < 1 \). Then by the triangle inequality there exists \( y'_1 \in \text{co}(L_1) \) such that \( \sup(\|y'_1 - y\|; y \in \overline{\text{co}}(L_1)) \leq h\delta(L_1) \).

Since \( y'_1 \) is a finite linear combination of members in \( L_1 \), there exists a \( c \)-subsequence \( \{y_n\} \) of \( \{x_n\} \) such that \( \sup(\|y_n - y\|; y \in \text{co}(L_{p_n})) \leq h\delta(L_{p_n}) \leq h\delta(L_1) \), and this inequality shows that \( \delta(\{y_n\}) \leq h\delta(L_1) \). By repeating the argument, there exists a successive \( c \)-subsequence with diameter less than or equal to \( h^n_1\delta(L_1) \). We need only repeat the argument a sufficient number \( k \) of times with \( h^{k+1}_1\delta(L_1) < \epsilon \) to obtain a contradiction.

Next by the diagonal method, there exists a \( c \)-subsequence of \( \{x_n\} \) which is norm Cauchy, and hence convergent to some \( y \). Then \( y \in K_n \).

Remarks. Bynum [1] showed that a uniformly convex Banach space has a uniformly normal structure. But the converse is not true. For example, the space \( l_2 \), renormed by

\[
\| (x_j) \|_1 = \max \left( |x_1|, \left( \sum_{j=2}^{\infty} |x_j|^2 \right)^{1/2} \right)
\]

has a uniformly normal structure, but \( (l_2, \| \cdot \|_1) \) is not uniformly convex.

Bynum [1] also showed that there exists a reflexive space with normal structure, but without a uniformly normal structure.

References


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