A NOTE ON AN OSCILLATION CRITERION FOR AN EQUATION WITH DAMPED TERM

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Abstract. A new oscillation criterion is given for the equation \( x''(t) + p(t)x'(t) + q(t)x(t) = 0, \quad t \in [t_0, \infty), \) where \( p(t) \) and \( q(t) \) are allowed to change sign on \([t_0, \infty)\).

Let us consider the second order differential equation with damped term

\[
(1) \quad x''(t) + p(t)x'(t) + q(t)x(t) = 0,
\]

and the more general equation

\[
(1)' \quad x''(t) + p(t)x'(t) + q(t)f(x(t)) = 0,
\]

where \( p, q \in C[t_0, \infty) \) and are allowed to assume negative values for arbitrarily large \( t, f \in C(R), \) \( xf(x) > 0 \) for \( x \neq 0. \)

We shall restrict our attention to solutions of (1) or (1)' which exist on some ray \([t, \infty)\). A solution of an equation is called oscillatory if it has no largest zero; otherwise it is nonoscillatory. An equation is said to be oscillatory if every solution is oscillatory.

For the second order linear differential equation

\[
(*) \quad x''(t) + q(t)x(t) = 0,
\]

Wintner [6] proved that a sufficient condition for oscillation was

\[
(**) \quad \lim_{t \to \infty} \frac{1}{t} \int_{t_0}^{t} q(\tau) \, d\tau \, ds = \infty.
\]

Hartman [3] proved that the limit cannot be replaced by the upper limit in condition (**) and

\[
- \infty < \liminf_{t \to \infty} \frac{1}{t} \int_{t_0}^{t} q(\tau) \, d\tau \, ds < \limsup_{t \to \infty} \frac{1}{t} \int_{t_0}^{t} q(\tau) \, d\tau \, ds \leq \infty
\]

implies (*) is oscillatory.

Later, important developments by Willett and Coles in averaging techniques for oscillation of (*) were made. Willett [5] and Coles [2], respectively, established more general theorems by considering weighted averages of the integral of \( q. \)

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Several years ago Kamenev [4] obtained an oscillation criterion for \((\star)\), namely, \((\star)\) is oscillatory if for some \(n > 2\),
\[
\lim_{t \to \infty} \sup_{t_0} t^{1-n} \int_{t_0}^{t} (t-s)^{n-1} q(s) \, ds = \infty,
\]
which extended Wintner’s result.

Recently, Yeh [8] has shown some oscillation criteria of \((1)'\) by using a technique similar to Kamenev’s, which included results of [1, 4 and 6].

The purpose of this note is to proceed further in this direction and present a new oscillation theorem which improves Kamenev’s criterion. A more general version of the theorem contains the theorems of Yeh [7 and 8].

Our result is as follows:

**Theorem.** Suppose for some \(\alpha \in (1, \infty)\) and \(\beta \in [0, 1)\),
\[
(2) \quad \lim_{t \to \infty} \sup_{t_0} t^{-\alpha} \int_{t_0}^{t} (t-s)^{-\alpha \beta} q(s) \, ds = \infty,
\]
\[
(3) \quad \lim_{t \to \infty} \sup_{t_0} t^{-\alpha} \int_{t_0}^{t} [(t-s)p(s)s + \alpha s - \beta(t-s)]^2(t-s)^{-\alpha \beta - 2} \, ds < \infty.
\]
Then \((1)\) is oscillatory.

**Proof.** Assume the contrary. Then \((1)\) has a nonoscillatory solution \(x(t)\). Without loss of generality, we may assume \(x(t) \neq 0\) for \(t \geq t_0\). Define \(\omega(t) = x'(t)/x(t)\). Then it follows from \((1)\) that
\[
\omega'(t) + \omega^2(t) + p(t)\omega(t) + q(t) = 0.
\]
Hence
\[
\int_{t_0}^{t} (t-s)^{-\alpha \beta} \omega'(s) \, ds + \int_{t_0}^{t} (t-s)^{-\alpha \beta} \omega^2(s) \, ds
\]
\[
+ \int_{t_0}^{t} (t-s)^{-\alpha \beta} p(s) \omega(s) \, ds + \int_{t_0}^{t} (t-s)^{-\alpha \beta} q(s) \, ds \leq 0.
\]

Noting that
\[
\int_{t_0}^{t} (t-s)^{-\alpha \beta} \omega'(s) \, ds = \alpha \int_{t_0}^{t} (t-s)^{-\alpha - 1} \beta \omega(s) \, ds - \beta \int_{t_0}^{t} (t-s)^{-\alpha \beta - 1} \omega(s) \, ds
\]
\[
- \omega(t_0)(t-t_0)^{-\alpha \beta} t_0^\beta,
\]
we obtain
\[
\int_{t_0}^{t} (t-s)^{-\alpha \beta} q(s) \, ds \leq \omega(t_0)(t-t_0)^{-\alpha \beta} t_0^\beta - \int_{t_0}^{t} (t-s)^{-\alpha \beta} \omega^2(s) \, ds
\]
\[
- \int_{t_0}^{t} [(t-s)p(s)s + \alpha s - \beta(t-s)](t-s)^{-\alpha - 1} \beta \omega(s) \, ds.
\]
Dividing by $t^\alpha$ and taking the upper limit as $t \to \infty$, we get

\[
\limsup_{t \to \infty} t^{-\alpha} \int_{t_0}^{t} (t - s)^{\alpha} q(s) \, ds = \omega(t_0) t^\beta + \limsup_{t \to \infty} \frac{t^{-\alpha}}{4} \int_{t_0}^{t} \left[ \left( (t - s)s + \alpha s - \beta(t - s) \right) \left( (t - s)^{\alpha/2} s^{\beta/2} \omega(s) \right) \right. \\
- \liminf_{t \to \infty} t^{-\alpha} \int_{t_0}^{t} \left( (t - s)^{\alpha/2} s^{\beta/2} \omega(s) \right) \\
\left. + \frac{1}{2} \left( (t - s)s + \alpha s - \beta(t - s) \right) (t - s)^{(\alpha - 2)/2} s^{(\beta - 2)/2} \right) \, ds < \infty,
\]

which contradicts conditions (2) and (3). This completes the proof.

Let \( p(t) = 0 \). Then (3) is satisfied automatically. Thus we have

**Corollary 1.** Suppose for some \( \alpha \in (1, \infty) \) and \( \beta \in [0, 1) \), (2) is satisfied. Then (1) is oscillatory.

**Remark 1.** Corollary 1 improves and generalizes Kamenev's theorem [4].

From the proof of the theorem, we easily obtain the following extension to (1)'.

**Corollary 2.** Suppose \( f'(x) \) exists and \( f'(x) \geq k > 0 \) for some constant \( k \) and for all \( x \neq 0 \). If (2) and (3) hold, then (1)' is oscillatory.

Taking \( \alpha = n - 1 \), \( \beta = 0 \) in (2) and (3), we get

**Corollary 3.** Suppose (4) is satisfied. If

\[
(2)' \quad \limsup_{t \to \infty} t^{1-n} \int_{t_0}^{t} (t - s)^{n-1} q(s) \, ds = \infty
\]

and

\[
(3)' \quad \limsup_{t \to \infty} t^{1-n} \int_{t_0}^{t} \left[ (t - s) p(s) + (n - 1) \right] (t - s)^{n-3} ds < \infty
\]

for some \( n > 2 \) (not necessarily integral), then (1)' is oscillatory.

**Remark 2.** Corollary 3 includes Kamenev's [4] and Yeh's theorem [7 and 8].

As an example, the equation

\[
(5) \quad x''(t) + \frac{\sin t}{t^\mu} x'(t) + \frac{\cos t}{t^\nu} x(t) = 0, \quad 1 \leq \mu < \infty, \ 0 \leq \nu < 1.
\]

Taking \( \alpha = 2, \nu < \beta < 1 \), we easily verify that all conditions of our theorem are satisfied. Hence, (5) is oscillatory. However, each of the criteria in [4, 7 and 8] fail to apply to (5). On the other hand, (5) cannot be reduced to a form in which some other known results may be used.

We could establish corresponding theorems by the method that is used in this note, which would improve other results of [8].

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REFERENCES


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