STABILITY OF TYPICAL CONTINUOUS FUNCTIONS
WITH RESPECT TO SOME
PROPERTIES OF THEIR ITERATES

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Abstract. Let \( I \) be a real compact interval, and let \( C \) be the space of continuous functions \( I \to I \) with the uniform metric. For \( f \in C \) denote \( \nu(f) = \sup_{n \to \infty} (\limsup_{n \to \infty} f^n(x) - \liminf_{n \to \infty} f^n(x)) \), where \( f^n \) is the \( n \)th iterate of \( f \). Then for each positive \( d \) there is an open set \( C^* \) dense in \( C \) such that the oscillation of \( \nu \) at each point of \( C^* \) is less than \( d \). Consequently, \( \nu \) is continuous in \( C \) except of the points of a first Baire category set.

Let \( I \) be a compact real interval, \( C \) the metric space of continuous functions \( I \to I \) with the uniform metric. For \( f \in C \) let \( \|f\| = \max \{f(x); x \in I\} \), and let \( f^n \) denote the \( n \)th iterate of \( f \). If \( f \) has a cycle \( x_1 \mapsto x_2 \mapsto \cdots \mapsto x_m \mapsto x_1 \), where \( x_i \neq x_j \) for \( i \neq j, i, j = 1, \ldots, m \), then the order of this cycle is \( m \) while the width of this cycle is \( \max(x_i - x_j) \). Let \( \lambda(f) \) be the l.u.b. of the widths of all cycles of \( f \), and put

\[
\nu(f) = \sup_{x \in I} \left( \limsup_{n \to \infty} f^n(x) - \liminf_{n \to \infty} f^n(x) \right).
\]

Clearly \( 0 \leq \lambda(f) \leq \nu(f) \) for each \( f \in C \), and \( \nu(f) = 0 \) iff \( f \) has no cycles (cf. [4]). The function \( \nu \) can be sometimes used as a "measure of chaos". Namely, if \( f \in C \) has no cycles, then each neighbourhood of \( f \) contains a chaotic function (cf. [2]). But if additionally the set of fixed points of \( f \) contains no interval, then for each \( g \) sufficiently near to \( f \), \( \nu(g) \) is small (although it can be positive) (cf. [5]).

L. Block [1] recently has shown that continuous functions are stable with respect to the order of their cycles: if \( f \in C \) has a cycle of order \( m \), and if \( n \) is greater than \( m \) in the Šarkovskii ordering (cf. [3 or 6]), then each \( g \in C \) sufficiently near to \( f \) has some \( n \)-cycle. However, Block’s result gives no information on the width of the corresponding cycles. As we show (see Theorem 1) the width of the cycles of functions \( g \) from a certain neighbourhood of \( f \) can be essentially smaller then the width of cycles of \( f \), i.e. \( \lambda(g) \ll \lambda(f) \), and in application the cycles of \( g \) then cannot be distinguished from the noise. The main aim of this paper is to show that such a type of behaviour is untypical. We begin with the following example.

Theorem 1. Given a \( \delta > 0 \) there is a continuous \( f \colon [0, 1] \to [0, 1] \) with the following properties: \( \lambda(f) = 1 \) and for each \( \varepsilon > 0 \) there is some continuous \( g \colon [0, 1] \to [0, 1] \) such that \( \|f - g\| < \varepsilon \) and \( \nu(g) < \delta \).
Proof. Let \( I_1, I_2, \ldots, I_{2n+1} \) be pairwise disjoint closed subintervals of \([0, 1]\) with the natural ordering

\[
I_{2n+1} < I_{2n-1} < \cdots < I_1 < I_2 < I_4 < \cdots < I_n
\]

(i.e. \( x < y \) for each \( x \in I_3 \) and \( y \in I_1 \), etc.). Put \( I_i = [x_i, y_i] \) for each \( i \), and assume \( x_{2n+1} = 0, \ y_{2n+1} = 1 \). Define a function \( f: [0, 1] \to [0, 1] \) in the following way: \( f(I_i) = I_{i+1} \) if \( i \neq 2n + 1 \) and \( f(I_{2n+1}) = I_1 \). Moreover, \( f \) is linear and decreasing on each \( I_i \), where \( i \neq 2n + 1 \), and linear and increasing on \( I_{2n+1} \). Finally, extend \( f \) continuously onto the whole \([0, 1]\).

It is easy to verify that \( f: x \mapsto y \mapsto x \mapsto y \mapsto \cdots \mapsto y \mapsto x \mapsto y \mapsto x \), i.e. \( x \) generates a cycle of order \((2n + 1)\) and of the width 1. Similarly \( y \) is a point of a \((2n + 1)\)-cycle, and from the linearity of \( f \) on each \( I_i \) we have \( f^{2n+1} \) is the identity mapping on \( I_1 \), and consequently, on each \( I_i \). Thus each point of \( I_1 \) is a cyclic point of order \( 2n + 1 \).

Now define a continuous \( g: [0, 1] \to [0, 1] \) such that \( g(x) = f(x) \) for \( x \in I_{2n+1} \), \( g(x_{2n+1}) = x_1 + \varepsilon \), \( g(y_{2n+1}) = y_1 \), and \( g \) is linear on \( I_{2n+1} \). Without loss of generality we may assume that \( x_1 + \varepsilon < y_1 \).

Clearly for each \( x \in I_{2n+1} \), except of \( x = y_{2n+1} \), we have \( f(x) < g(x) = f(y) \) for some \( y \in I_{2n+1}, y > x \). Hence

\[
g^{2n+1}(x) = f^{2n+1}(y) = y > x.
\]

Thus \( g \) has in \( I_{2n+1} \) exactly one cyclic point of order \( > 1 \), namely the point \( y_{2n+1} \).

To finish the proof note that when the length of \( I_{2n+1} \) is greater than \( 1 - \delta \), then we have \( \lambda(g) < \delta \). Moreover, by (1) the cycle of \( g \) generated by \( y_{2n+1} \) attracts all points of \( I_{2n+1} \), and this implies \( \nu(g) < \delta \). Q.E.D.

Remark. The theorem holds when \( 2n + 1 \) is replaced by any \( m > 1 \). Also Theorem 1 shows that neither \( \nu \) nor \( \lambda \) is lower semicontinuous. An example exhibiting that these functions are not upper semicontinuous can be found in [5].

In the proof of our main result the following lemmas are useful.

Lemma 1. Let \( f \in C \) with \( \nu(f) > d \). Then for each \( \varepsilon > 0 \) there is some \( g \in C \) such that \( \| f - g \| < \varepsilon \) and \( g \) has a cycle of the width greater than \( d \).

Proof. Choose \( x \in I \) such that

\[
\limsup_{n \to \infty} f^n(x) - \liminf_{n \to \infty} f^n(x) > d.
\]

For simplicity denote \( f^n(x) = x_n \). Without loss of generality we may assume that the sequence \( \{x_n\} \) is not periodic. Let \( \delta > 0 \) be such that \( |f(u) - f(v)| < \varepsilon \) whenever \( |u - v| < \delta \) for \( u, v \in I \). By (2) there are indexes \( n(1) < n(2) < n(3) \) such that

\[
| x_{n(1)} - x_{n(3)} | < \delta
\]

and

\[
| x_{n(2)} - x_{n(3)} | > d.
\]

Now let \( g(x_n) = f(x_n) \) for \( n = n(1) + 1, \ldots, n(3) - 1 \), \( g(x_{n(3)}) = f(x_{n(1)}) \) and let \( g \) be continuous in \( I \). Moreover, by (3), \( | g(x_{n(3)}) - f(x_{n(3)}) | < \varepsilon \) hence we can choose \( g \).
such that \( \|f - g\| < \varepsilon \). It is easy to verify that \( g \) has a cycle \( x_{n(1)} \mapsto x_{n(1) + 1} \mapsto \cdots \mapsto x_{n(3)} \mapsto x_{n(1) + 1} \) of order \( k = n(3) - n(1) \). By (4) the width of this cycle is greater than \( d \). Q.E.D.

**Lemma 2.** Assume that \( f \in C \) and that \( \lambda(f) > d \). Then for each \( \varepsilon > 0 \) there is some \( g \in C \) with the following properties: \( \|f - g\| < \varepsilon \) and for each \( h \in C \) from a sufficiently small neighbourhood of \( g \) we have \( \lambda(h) > d \).

**Proof.** Choose a cycle \( x_1 \mapsto x_2 \mapsto \cdots \mapsto x_n \mapsto x_1 \) of the function \( f \) of the width \( d_1 > d \). First assume that one of the points \( x_i \), say \( x_1 \), is an interior point of \( I \). Choose also \( \delta \) with \( 0 < \delta < (d_1 - d)/2 \) such that for every \( u, v \in I \),

\[
|u - v| < \delta \quad \text{implies} \quad |f(u) - f(v)| < \varepsilon.
\]

Moreover, let \( U_i \) be an open interval containing \( x_i \) and denote \( U_i = f^{i-1}(U_i) \), \( i = 2, \ldots, n \). Without loss of generality we may assume that \( U_i \) is so small that the sets \( U_i \) are pairwise disjoint, and that the length of \( U_1 \) is less than \( \delta \). Now choose \( u, v \in U_1 \) such that \( u < x_1 < v \). Define \( g \) by \( g(y) = f(y) \) for \( y \notin U_1 \), \( g(u) = g(v) = x_2 \), and extend \( g \) continuously to a function \( I \rightarrow I \). By (5) \( g \) can be chosen such that \( \|f - g\| < \varepsilon \). We have

\[
g^n(u) = x_1 > u \quad \text{and} \quad g^n(v) = x_1 < v.
\]

If \( h \in C \) is near \( g \), then \( h((u, v)) \subset U_{i+1}, \ i = 1, \ldots, n - 1 \), and by (6), \( h^n(u) > u \), \( h^n(v) < v \). Hence \( h \) has a periodic point \( \xi \in (u, v) \). The order of \( \xi \) must be \( n \) since \( h^i(\xi) \in U_{i+1}, \ i = 1, \ldots, n - 1 \). Also it is easy to see that the width of this cycle is greater than \( d_1 - 2\delta > d \).

It remains to consider the case when \( n = 2 \) and \( x_1, x_2 \) are the endpoints of \( I \). Assume that \( x_1 < x_2 \) and define \( g \) by \( g(x) = x_2 \) for \( x \in [x_1, x_1 + \mu] \) and \( g(x) = x_1 \) for \( x \in [x_2 - \mu, x_2] \), where \( \mu > 0 \) is small. Moreover, let \( g(x) = f(x) \) for \( x \in [x_1 + 2\mu, x_2 - 2\mu] \), and let \( g \) be continuous in \( I \). Clearly for \( \mu \) sufficiently small \( g \) can be chosen such that \( \|f - g\| < \varepsilon \). Also assume that \( x_2 - x_1 - 2\mu > d \). Now let \( h \in C \) with \( \|h - g\| < \mu/2 \). Then \( h^2(x_1 + \mu) = x_1 < x_1 + \mu \). On the other hand, since \( h \): \( I \rightarrow I \), we have \( h(x_1) \geq x_1 \). Thus for some \( \xi \in [x_1, x_1 + \mu] \), \( h^2(\xi) = \xi \) and clearly \( h(\xi) \neq \xi \). It is easy to see that \( |h(\xi) - \xi| < x_2 - x_1 - 2\mu > d \), i.e. \( \lambda(h) > d \), and the lemma is proved.

In the following the oscillation \( \omega_\varepsilon(f) \) of \( f \) at \( f \in C \) is defined by

\[
\omega_\varepsilon(f) = \limsup |\nu(g) - \nu(f)| \quad \text{for} \quad \|g - f\| \rightarrow 0, \ g \in C,
\]

and similarly we define \( \omega_\lambda(f) \). Now we are able to give the main result.

**Theorem 2.** Let \( \delta > 0 \). Then there is a subset \( C_\delta \) of \( C \), which is nowhere dense in \( C \) and such that the oscillation both of \( \nu \) and \( \lambda \) at each \( f \in C \setminus C_\delta \) is less than \( \delta \).

In other words, when \( \delta \) is small, both functions \( \nu \) and \( \lambda \) are continuous in the points of the set \( C \setminus C_\delta \) up to small perturbations.

**Corollary.** There is a first Baire category set \( K \subset C \) such that both \( \nu \) and \( \lambda \) are continuous in the points of the set \( C \setminus K \).
Proof. Put $K = \bigcup_{n=1}^{\infty} C_1/n$.

Proof of Theorem 1. Fix some positive integer $n$. Let $A_i = \{ f \in C; \nu(f) > i/n \}$ for $i = 1, \ldots, n$, and put $A_0 = C$, $A_{n+1} = \emptyset$. Let $B_i$ be the boundary of $A_i$, i.e. $B_i = \text{Clos} A_i \cap \text{Clos}(C \setminus A_i)$, where Clos is the closure operator in $C$. Then each $B_i$ is closed and nowhere dense in $C$. To see the second property, let $G$ be an open set. It suffices to consider the case $G \cap A_i \neq \emptyset$, where $i \neq 0$. By Lemma 1 there is some $f \in G$ with $\lambda(f) > i/n$, and by Lemma 2 there is a nonempty open subset $H \subset G$ such that $\lambda(h) > i/n$, and hence $\nu(h) > i/n$ for each $h \in H$. Thus $H \cap B_i = \emptyset$.

Now put $B = B_1 \cup \cdots \cup B_n$. Let $f \in C \setminus B$ and let $j$ be the greatest index with $f \in A_j$. Then $f$ is an interior point of $A_j$ and $f$ does not belong to the boundary of $A_{j+1} \subset A_j$; hence $f \in A_j \setminus \text{Clos} A_{j+1}$. Now it is easy to see that $\omega_f(f) \leq 1/n$.

Similarly we can choose a nowhere dense set $D \subset C$ such that $\omega_f(f) \leq 1/n$ for each $f \in C \setminus D$. To finish the proof take $n$ such that $1/n < \delta$ and put $C_\delta = B \cup D$.

Remark. There is an open problem, whether for some $f$, $\lambda(f) < \nu(f)$. We conjecture that the answer is positive, but we have no example of such a function.

References


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