A "RIEMANN HYPOTHESIS" FOR TRIANGULABLE MANIFOLDS

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Abstract. Given a triangulable manifold we show how to find a triangulation whose characteristic polynomial has roots which are either real or on the line \( \Re z = 1/2 \).

If \( K \) is a (finite) simplicial complex, then \( f_K(z) \) will denote the polynomial
\[
\frac{\chi}{2} - f_0(K) \cdot z + f_i(K) \cdot z^2 - \cdots; \quad \text{here } \chi \text{ is the Euler characteristic of the underlying space } M = |K| \text{ and } f_i(K) \text{ is the number of } i\text{-simplices in } K.
\]

Theorem. If \( M \) is any closed triangulable manifold, then it admits a triangulation \( K \) for which all the nonreal zeros of \( f_K(z) \) lie on the line \( \Re z = 1/2 \).

Proof. If \( L \) is any triangulation of \( M^m \), then one has the functional equation
\[
f_L(z) = (-1)^{m+1} f_L(1 - z). \quad \text{(This fact is well known and is a concise way of writing the Dehn-Sommerville equations (see e.g. [1, p. 101]): it was observed by Klee [2] that these equations hold if the link of each } i\text{ -simplex of } L \text{ has the same Euler characteristic as an } (m - i - 1)\text{-dimensional sphere, e.g. if } L \text{ triangulates a closed } m\text{-manifold.) So the roots of } f_L(z) \text{ are symmetrically situated about the real axis and the line } \Re z = 1/2. \]

For each integer \( q \geq 0 \) we construct a simplicial complex \( L_q \) as follows: \( L_0 = L \) is any triangulation of \( M^m \) and \( L_{q+1} \) is obtained by deriving an \( m \)-simplex of \( L_q \). We note that
\[
\begin{align*}
\frac{f_{L_q}(z)}{f_L(z)} &= f_{L_q}(z) - qz + q(m + 1)z^2 - q(m + 1)z^3 + \cdots \\
&= f_L(z) - qz(1 - z)^{m+1} - (-1)^{m+1} qz^{m+1} \left(1 - z\right).
\end{align*}
\]

We assert that for all \( q \) sufficiently big \( K = L_q \) is a triangulation of \( M^m \) such that \( f_K(z) \) has distinct roots of which all but 2 lie on the line \( \Re z = 1/2 \). It is clear that the remaining 2 roots must then be equal to \( 1/2 \pm \kappa \) for some \( \kappa > 0 \); if \( \chi = 0 \) these exceptional roots are obviously 0 and 1.

Note that \( f_K(1 - z) = (-1)^{m+1} f_K(z) \) and \( f_K(\bar{z}) = f_K(z) \) imply that for \( m \) odd (resp. \( m \) even) \( f_K(z) \) takes real (resp. purely imaginary) values on the line \( \Re z = 1/2 \); the same is also true for the degree \( m + 1 \) polynomial
\[
-z(1 - z)^{m+1} - (-1)^{m+1} z^{m+1} (1 - z) = q^{-1} f_K(z) - q^{-1} \cdot f_L(z).
\]
Next we observe that the $m - 1$ roots of $-z(1 - z)^{m+1} - (-1)^{m+1}z^m(1 - z)$ other than 0 and 1 satisfy $|z/(1 - z)| = 1$, i.e. lie on the line $\text{Re} z = 1/2$. So for $q$ big the neighbouring polynomial $q^{-1}f_k(z)$ must also have $m - 1$ roots on the line $\text{Re} z = 1/2$. Q.E.D.

**Remark.** Let $L$ be a triangulation of $M^n$ and let $C(q, m + 1)$, $q \geq m + 2$, be a cyclic triangulation (see e.g. [1, p. 82]) of the sphere $S^m$. By omitting an $m$-simplex each from $L$ and $C(q, m + 1)$ and then identifying their boundaries, one gets a triangulation $L^q$ of $M^m$. One can verify (using equation (13) on p. 172 of [1]) to examine the roots of the polynomial of $C(q, m + 1))$ that if $m \geq 5$ and $q$ is sufficiently big, then $f_{L^q}(z)$ has some roots which are neither real nor on the line $\text{Re} z = 1/2$.

The “Riemann hypothesis” considered above is related to the lower and upper bound conjectures for manifolds and is amongst the problems posed in §6 of [3].

I am grateful to the referee for pointing out a mistake in the original version of this paper.

**References**


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