A "RIEMANN HYPOTHESIS" FOR TRIANGULABLE MANIFOLDS

K. S. SARKARIA

Abstract. Given a triangulable manifold we show how to find a triangulation whose characteristic polynomial has roots which are either real or on the line $Re z = 1/2$.

If $K$ is a (finite) simplicial complex, then $f_K(z)$ will denote the polynomial $\chi/2 - f_0(K)z + f_1(K)z^2 - \cdots$; here $\chi$ is the Euler characteristic of the underlying space $M = |K|$ and $f_i(K)$ is the number of $i$-simplices in $K$.

Theorem. If $M$ is any closed triangulable manifold, then it admits a triangulation $K$ for which all the nonreal zeros of $f_K(z)$ lie on the line $Re z = 1/2$.

Proof. If $L$ is any triangulation of $M^m$, then one has the functional equation $f_L(z) = (-1)^m+1f_L(1 - z)$. (This fact is well known and is a concise way of writing the Dehn-Sommerville equations (see e.g. [1, p. 101]): it was observed by Klee [2] that these equations hold if the link of each $i$-simplex of $L$ has the same Euler characteristic as an $(m - i - 1)$-dimensional sphere, e.g. if $L$ triangulates a closed $m$-manifold.) So the roots of $f_L(z)$ are symmetrically situated about the real axis and the line $Re z = 1/2$.

For each integer $q \geq 0$ we construct a simplicial complex $L_q$ as follows: $L_0 = L$ is any triangulation of $M^m$ and $L_{q+1}$ is obtained by deriving an $m$-simplex of $L_q$. We note that

$$f_{L_q}(z) = f_L(z) - qz + q(m + 1)z^2 - q\left(\frac{m + 1}{2}\right)z^3 + \cdots$$

$$+ (-1)^{m+1}q\binom{m + 1}{m}z^{m+1} - (-1)^{m+1}qz^{m+1}$$

$$= f_L(z) - qz(1 - z)^{m+1} - (-1)^{m+1}qz^{m+1}(1 - z).$$

We assert that for all $q$ sufficiently big $K = L_q$ is a triangulation of $M^m$ such that $f_K(z)$ has distinct roots of which all but 2 lie on the line $Re z = 1/2$. It is clear that the remaining 2 roots must then be equal to $1/2 \pm \kappa$ for some $\kappa > 0$; if $\chi = 0$ these exceptional roots are obviously 0 and 1.

Note that $f_K(1 - z) = (-1)^{m+1}f_K(z)$ and $f_K(\bar{z}) = \overline{f_K(z)}$ imply that for $m$ odd (resp. $m$ even) $f_K(z)$ takes real (resp. purely imaginary) values on the line $Re z = 1/2$; the same is also true for the degree $m + 1$ polynomial

$$-z(1 - z)^{m+1} - (-1)^{m+1}z^{m+1}(1 - z) = q^{-1}f_K(z) - q^{-1}\cdot f_L(z).$$

Received by the editors September 3, 1982 and, in revised form, June 6, 1983.

1980 Mathematics Subject Classification. Primary 57Q15; Secondary 52A40, 05C15.

©1984 American Mathematical Society

0002-9939/84 $1.00 + $.25 per page
Next we observe that the \( m - 1 \) roots of \( -z(1 - z)^m - (-1)^m(1 - z) \) other than 0 and 1 satisfy \( |z/(1 - z)| = 1 \), i.e. lie on the line \( \text{Re} \ z = 1/2 \). So for \( q \) big the neighbouring polynomial \( q^{-1} f_K(z) \) must also have \( m - 1 \) roots on the line \( \text{Re} \ z = 1/2 \). Q.E.D.

**Remark.** Let \( L \) be a triangulation of \( M^n \) and let \( C(q, m + 1), q \geq m + 2 \), be a cyclic triangulation (see e.g. [1, p. 82]) of the sphere \( S^m \). By omitting an \( m \)-simplex each from \( L \) and \( C(q, m + 1) \) and then identifying their boundaries, one gets a triangulation \( L^q \) of \( M^n \). One can verify (using equation (13) on p. 172 of [1]) to examine the roots of the polynomial of \( C(q, m + 1) \) that if \( m \geq 5 \) and \( q \) is sufficiently big, then \( f_{L^q}(z) \) has some roots which are neither real nor on the line \( \text{Re} \ z = 1/2 \)

The “Riemann hypothesis” considered above is related to the lower and upper bound conjectures for manifolds and is amongst the problems posed in §6 of [3].

I am grateful to the referee for pointing out a mistake in the original version of this paper.

**REFERENCES**


213, 16A, Chandigarh 160016, India