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ON THE SEMISIMPLICITY OF SKEW POLYNOMIAL RINGS

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ABSTRACT. Let R be a ring satisfying the maximal condition on annihilator left ideals and σ be an automorphism of R. We show that the Jacobson radical of the skew polynomial ring $R_\sigma[x]$ is nonzero if and only if the prime radical of $R_\sigma[x]$ is nonzero. Furthermore, it is so if and only if the prime radical of R is nonzero. In general, an example is given of a commutative semisimple algebra R and an automorphism σ such that $R_\sigma[x]$ is prime but the Levitzki radical of $R_\sigma[x]$ is nonzero.

0. Introduction. Let R be a ring and σ an automorphism of R. In this paper we study the semisimplicity of the skew polynomial ring $R_\sigma[x]$, and that of the skew group ring $R_\sigma\langle x\rangle$ of the infinite cyclic group.

If R satisfies the maximal chain condition on annihilator left ideals, then we prove that the conditions: (i) $J(R_\sigma\langle x\rangle)\neq 0$, (ii) $J(R_\sigma[x])\neq 0$, (iii) $P(R)\neq 0$, (iv) $P(R_\sigma\langle x\rangle)\neq 0$, (v) $P(R_\sigma[x])\neq 0$, are equivalent (Theorem 2.1). In general, we construct a commutative semisimple algebra R and an algebra automorphism σ where we prove, using Van der Waerden's result about arbitrary long arithmetic progressions, that the Levitzki radical $\mathfrak{L}(R_\sigma[x])$ of $R_\sigma[x]$ is nonzero but the ring $R_\sigma[x]$ is prime.

Finally, we discuss the semisimplicity question for $R_\sigma[x]$, where R is an arbitrary commutative ring. In the example (Example 3.2) R is a commutative ring and $\mathfrak{L}(R_\sigma[x])\neq 0$, but $P(R_\sigma[x])=0$. When R is a commutative ring, we raise the question: $J(R_\sigma[x])\neq 0$ if and only if $\mathfrak{L}(R_\sigma[x])\neq 0$. We have settled this question in the affirmative in some cases.

1. Definitions and preliminaries.

1.1. Let R be a ring with 1 and σ be an automorphism of R. Then by the skew polynomial ring $R_\sigma[x]$, we mean the ring

$$R_\sigma[x]=\left\{\sum_{i\geqslant 0} r_i x^i\colon r_i\in R \text{ and almost all } r_i \text{ are zero}\right\}$$

with addition componentwise and multiplication defined by the rule $xr=\sigma(r)x$. By the skew group ring $R_\sigma\langle x\rangle$, we mean the ring

$$R_\sigma\langle x\rangle=\left\{\sum_{i\in Z} r_i x^i \,\middle|\, r_i\in R \text{ and almost all } r_i \text{ are zero}\right\}$$

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with addition componentwise and multiplication defined by $x^i r = \sigma^i(r)x^i$.

1.2. An element $r \in R$ is said to be *σ-nilpotent* if for all natural integers m there exists an integer $n = n(m) > 2$, depending on m, such that

$$r\sigma^m(r) \cdots \sigma^{m(n-1)}(r) = 0.$$

It is said to be *σ-nilpotent of bounded index* if we can find a common integer n such that the above equality is satisfied for all natural integers m, and the least such n is called the *σ-nilpotency index of r. An ideal I of R is said to be σ-nil* if $\sigma(I) = I$ and every element of I is σ-nilpotent.

2. Main results. In Theorem 3.1 of [**1**] by Bedi and the author, it is proved that

$$J(R_\sigma[x]) = I \cap J(R) + Ix + Ix^2 + \cdots + Ix^n + \cdots,$$

where $I = \{r \in R\colon rx \in J(R_\sigma[x])\}$. Here $\sigma(I) = I$. Further, for all $r \in I$, $rx^n \in J(R_\sigma[x])$, and so rx^n is nilpotent by [**1**, Lemma 2.4]. This proves that I is σ-nil ideal of R. Hence if $J(R_\sigma[x]) \neq 0$ then there exists a nonzero σ-nil ideal. We will use this result without any further mention.

THEOREM 2.1. *Let R be a ring satisfying the maximal condition on annihilator left ideals and σ be an automorphism of R. Then the following are equivalent*:

(1) $J(R_\sigma\langle x\rangle)$ *is nonzero.*

(2) $J(R_\sigma[x])$ *is nonzero.*

(3) *R has a nonzero σ-nil ideal.*

(4) *R has a nonzero right nil ideal.*

(5) $P(R)$ *is nonzero.*

(6) $P(R_\sigma\langle x\rangle)$ *is nonzero.*

(7) $P(R_\sigma[x])$ *is nonzero.*

PROOF. To prove the theorem we first show that (1) ⇒ (2) ⇒ (3) ⇒ (4) ⇒ (5) ⇒ (6) ⇒ (1), and then we show that (6) ⇒ (7) ⇒ (2).

(1) ⇒ (2). This result follows from [**1**, Theorem 3.1].

(2) ⇒ (3). If $J(R_\sigma[x])$ then the ring R has a nonzero σ-nil ideal.

(3) ⇒ (4). Let I be a nonzero σ-nil ideal of R. Assume I is not nil and choose r in I such that r is not nilpotent. We claim that there exists $n \geqslant 1$ such that $r\sigma^n(r) \neq 0$. Let us suppose that $r\sigma^n(r) = 0$ for all natural integers n. Define

$$I_m = \sigma^m(r)R + \sigma^{m+1}(r)R + \cdots + \cdots$$

for all natural integers m. Clearly

$$I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots \supseteq \cdots.$$

Thus

$$\mathrm{Ann}_l(I_1) \subseteq \mathrm{Ann}_l(I_2) \subseteq \mathrm{Ann}_l(I_3) \subseteq \cdots \subseteq \cdots.$$

Let $\mathrm{Ann}_l(I_t) = \mathrm{Ann}_l(I_{t+1})$. Since $r\sigma^n(r) = 0$ for all natural integers n, $\sigma^t(r) \in \mathrm{Ann}_l(I_{t+1}) = \mathrm{Ann}_l(I_t)$. Thus

$$\sigma^t(r)I_t = 0 \Rightarrow \sigma^t(r)\sigma^t(r) = 0 \Rightarrow r^2 = 0.$$

Thus, there exists $n \geqslant 1$ such that $r\sigma^n(r) \neq 0$. However $r\sigma^n(r) \cdots \sigma^{tn}(r) = 0$ for some $t > 1$. We assume that t is minimal. Put $s = r\sigma^n(r) \cdots \sigma^{(t-1)n}(r)$. If the right ideal sR is not nil, sk is not nilpotent for some k. Put $r_1 = r$ and $r_2 = sk$. We show that $\mathrm{Ann}_l(r_1) \subsetneqq \mathrm{Ann}_l(r_2)$. Since $r\sigma^n(r) \cdots \sigma^{tn}(r) = 0$, $\sigma^{-n}(r) \in \mathrm{Ann}_l(r_2)$. But $\sigma^{-n}(r) \notin \mathrm{Ann}_l(r_1)$. Thus $\mathrm{Ann}_l(r_1) \subsetneqq \mathrm{Ann}_l(r_2)$. If r_2R is not nil, then arguing as before we get r_3 such that $\mathrm{Ann}_l(r_1) \subsetneqq \mathrm{Ann}_l(r_2) \subsetneqq \mathrm{Ann}_l(r_3)$. Continuing in this fashion we will get a nonzero nil right ideal.

(4) $\Rightarrow$ (5). This follows from [**6**, Lemma 4.7].

(5) $\Rightarrow$ (6). Let

$$\mathfrak{F} = \left\{\mathrm{Ann}_l(Rx) \mid xRx = 0, x \neq 0\right\}.$$

Since $P(R) \neq 0$, $\mathfrak{F}$ is nonempty. Let $\mathrm{Ann}_l(Rr)$ be a maximal element of $\mathfrak{F}$. We first show that $rR\sigma^n(r) = 0$ for all integers n. If it is not true, then $r\sigma^m(ar) \neq 0$ for an integer m. Let $s = \sigma^{-m}(r)ar$. Clearly $sRs = 0$ and $\mathrm{Ann}_l(Rs) \supseteq \mathrm{Ann}_l(Rr)$. This inclusion is strict because $\sigma^{-m}(r)Rr \neq 0$ and

$$\sigma^{-m}(r)Rs = \sigma^{-m}(r)R\sigma^{-m}(r)ar = 0.$$

Thus $rR\sigma^n(r) = 0$ for all integers n.

Now, observe that $rR_\sigma\langle x\rangle r = 0$. Hence $P(R_\sigma\langle x\rangle)$ is nonzero.

(6) $\Rightarrow$ (1). This is clear.

(6) $\Rightarrow$ (7). Since $P(R_\sigma\langle x\rangle) \neq 0$, let I be a nonzero nilpotent ideal of $R_\sigma\langle x\rangle$. Now, $I \cap R_\sigma[x]$ is a nonzero nilpotent ideal of $R_\sigma[x]$. Hence $P(R_\sigma[x]) \neq 0$.

(7) $\Rightarrow$ (2). This is clear.

3. An example. In this section, we give an example of a commutative semisimple algebra R and an automorphism σ such that $R_\sigma[x]$ is prime but the Levitzki radical of $R_\sigma[x]$ is nonzero. In the construction of the example we use the result:

Let $G(k, m)$ denote the least integer such that if $g \geqslant G(k, m)$ and if $A = \{a_n\}_{n=0}^{g-1}$ is a strictly increasing sequence of integers with bounded gaps $a_n - a_{n-1} \leqslant m$, $1 \leqslant n \leqslant g-1$, then A contains a k-term arithmetic progression. The number $G(k, m)$ does exist [**5**]. The existence of $G(k, m)$ is an easy consequence of Van der Waerden's theorem [**3**, **7**].

THEOREM 3.1. *Let R be a commutative ring and σ be an automorphism of R. If $r \in R$ is a nonzero σ-nilpotent element of bounded index then $rxR_\sigma[x]$ is a locally nilpotent right ideal of $R_\sigma[x]$.*

PROOF. Let $I = rxR_\sigma[x]$. Clearly I is a right ideal of $R_\sigma[x]$. To prove that I is locally nilpotent, it suffices to prove that the subring S generated by

$$T = \left\{rs_1x^{i_1}, rs_2x^{i_2}, \ldots, rs_nx^{i_n}\right\}$$

is nilpotent for all $s_j \in R$, $i_j \geqslant 1$. Let $m = \max(i_1, i_2, \ldots, i_n)$ and k be the σ-nilpotency index of r. Let $l = G(k, m)$. We claim that $S^l = 0$. For this, it suffices to prove that the product of any l monomials, each of the type given in T, is zero. Any typical

such product P is given by

$$P = \left(rs_{t_1}x^{i_{t_1}}\right)\left(rs_{t_2}x^{i_{t_2}}\right)\cdots\left(rs_{t_l}x^{i_{t_l}}\right)$$
$$= \left(r\sigma^{i_{t_1}}(r)\sigma^{i_{t_1}+i_{t_2}}(r)\cdots\sigma^{i_{t_1}+i_{t_{l-1}}}(r)\cdots\right)(\cdots)x^{i_{t_1}+i_{t_l}}.$$

Thus, it follows that $P = 0$, since the set

$$\{0, i_{t_1}, i_{t_1}+i_{t_2},\ldots,i_{t_1}+i_{t_2}+\cdots+i_{t_{l-1}}\}$$

contains k numbers in arithmetic progression, and k is the σ-nilpotency index of r.

Here we raise a question: Let R be a commutative ring. Is $J(R_\sigma[x])$ nonzero if and only if $\mathfrak{L}(R_\sigma[x])$ is nonzero? If $J(R_\sigma[x])$ is nonzero then R has a nonzero σ-nil ideal, say I, and hence Theorem 3.1 settles this question in the affirmative, in the case when I contains a σ-nilpotent element of bounded index.

EXAMPLE 3.2. Let K be an infinite field and S be a polynomial ring over K in the commuting indeterminates x_i indexed by the set of integers Z. Let σ be a map from S to S such that

$$\sigma(x_i) = x_{i+1} \quad \text{and} \quad \sigma(k) = k \quad \text{for all } k \in K.$$

Then σ is an automorphism of S. Consider the ideal I of S generated by the set $T = \{x_k x_{k+a} x_{k+2a}\colon\ k, a \in Z \text{ and } a \neq 0\}$. Since σ permutes the elements of T, $\sigma(\mathrm{I}) = I$. Let $R = S/I$. Also, σ induces an automorphism of R and we denote this automorphism again by σ. We can think of R as a k-linear span of the monomials

$$\left\{ x_{i_1}^{t_1}x_{i_2}^{t_2}\cdots x_{i_k}^{t_k}\colon \begin{array}{l} \text{no three elements of } i_1, i_2,\ldots,i_k \\ \text{are in arithmetic progression} \end{array} \right\}$$

with usual addition, and we multiply them by using the rule $x_k x_{k+a} x_{k+2a} = 0$ for all $k, a \in Z$ with $a \neq 0$.

Now, we prove the following: (i) R is semisimple; (ii) $R_\sigma[x]$ is prime; (iii) $\mathfrak{L}(R_\sigma[x])$ is nonzero.

(i) *R is semisimple*. First, we prove that if $J(R) \neq 0$ then $J(R)$ contains a monomial. If $0 \neq r \in J(R)$ then we write r as a polynomial in one of the indeterminates, say x_{n_1}; let

$$r = r_0 + r_1 x_{n_1} + \cdots + r_{t_1} x_{n_1}^{t_1}.$$

Let $\alpha_1, \alpha_2,\ldots,\alpha_{t_1+1}$ be distinct nonzero elements of K and consider the algebra automorphisms θ_i of R, given by

$$\theta_i(x_{n_1}) = \alpha_i x_{n_1} \qquad \theta_i(x_j) = x_j \quad \text{if } j \neq n_1.$$

Now $\theta_i(r) = r_0 + r_1\alpha_1 x_{n_1} + \cdots + r_{t_1}\alpha_i^{t_1} x_{n_1}^{t_1} \in J(R)$ for all i. These equations give that $r_{t_1}x_{n_1}^{t_1}$ is a K-linear combination of $\theta_1(r), \theta_2(r),\ldots,\theta_{t_1+1}(r)$, and thus, $r_{t_1}x_{n_1}^{t_1} \in J(R)$. Again, we write the element $r_{t_1}x_{n_1}^{t_1}$ as a polynomial in some other indeterminate and repeat the process. Continuing this way, we finally get that $0 \neq s = kx_{n_1}^{t_1}\cdots x_{n_m}^{t_m} \in J(R)$. Now, since R is a graded ring under the obvious grading and s is a monomial in $J(R)$, by [**1**, Lemma 2.4], s is nilpotent; however, this is not possible because no three elements of $n_1, n_2,\ldots,n_m$ are in arithmetic progression.

(ii) $R_\sigma[x]$ *is prime*. Let $f(x) = a_0 + a_1x + \cdots + a_nx^n$ and $g(x) = b_0 + b_1x + \cdots + b_mx^m$ with nonzero a_n and b_m such that

$$f(x)R_\sigma[x]g(x) = 0 \Rightarrow f(x)x^jg(x) = 0 \quad \text{for all } j \geqslant 0.$$

Comparing the coefficient of highest degree term we get that

$$a_n\sigma^{n+j}(b_m) = 0 \quad \text{for all } j \geqslant 0.$$

Thus, if $R_\sigma[x]$ is not prime, there exists $a = a_n$ and $b = b_m$ such that

$$a\sigma^k(b) = 0 \quad \text{for all } k \geqslant n.$$

However, this is not possible and this can be shown by selecting a sufficiently large integer k, so that the indeterminates involved in $\sigma^k(b)$ are disjoint from those involved in a, and the product of any two monomials in $a\sigma^k(b)$ does not contain three indeterminates whose suffixes are in arithmetic progression.

(iii) $\mathfrak{L}(R_\sigma[x])$ *is nonzero*. An element $x_1 \in R$ is σ-nilpotent of bounded index (σ-nilpotency index of x_1 is three) and hence by Theorem 3.1, $\mathfrak{L}(R_\sigma[x])$ is nonzero.

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