ON THE SEMISIMPLICITY OF SKEW POLYNOMIAL RINGS

JAI RAM

Abstract. Let \( R \) be a ring satisfying the maximal condition on annihilator left ideals and \( \sigma \) be an automorphism of \( R \). We show that the Jacobson radical of the skew polynomial ring \( R_\sigma[x] \) is nonzero if and only if the prime radical of \( R_\sigma[x] \) is nonzero. Furthermore, it is so if and only if the prime radical of \( R \) is nonzero. In general, an example is given of a commutative semisimple algebra \( R \) and an automorphism \( \sigma \) such that \( R_\sigma[x] \) is prime but the Levitzki radical of \( R_\sigma[x] \) is nonzero.

0. Introduction. Let \( R \) be a ring and \( \sigma \) an automorphism of \( R \). In this paper we study the semisimplicity of the skew polynomial ring \( R_\sigma[x] \), and that of the skew group ring \( R_\sigma \langle x \rangle \) of the infinite cyclic group.

If \( R \) satisfies the maximal chain condition on annihilator left ideals, then we prove that the conditions: (i) \( J(R_\sigma \langle x \rangle) \neq 0 \), (ii) \( J(R_\sigma[x]) \neq 0 \), (iii) \( P(R) \neq 0 \), (iv) \( P(R_\sigma \langle x \rangle) \neq 0 \), (v) \( P(R_\sigma[x]) \neq 0 \), are equivalent (Theorem 2.1). In general, we construct a commutative semisimple algebra \( R \) and an algebra automorphism \( \sigma \) where we prove, using Van der Waerden's result about arbitrary long arithmetic progressions, that the Levitzki radical \( \mathcal{E}(R_\sigma[x]) \) of \( R_\sigma[x] \) is nonzero but the ring \( R_\sigma[x] \) is prime.

Finally, we discuss the semisimplicity question for \( R_\sigma[x] \), where \( R \) is an arbitrary commutative ring. In the example (Example 3.2) \( R \) is a commutative ring and \( \mathcal{E}(R_\sigma[x]) \neq 0 \), but \( P(R_\sigma[x]) = 0 \). When \( R \) is a commutative ring, we raise the question: \( J(R_\sigma[x]) \neq 0 \) if and only if \( \mathcal{E}(R_\sigma[x]) \neq 0 \). We have settled this question in the affirmative in some cases.

1. Definitions and preliminaries.

1.1. Let \( R \) be a ring with 1 and \( \sigma \) be an automorphism of \( R \). Then by the skew polynomial ring \( R_\sigma[x] \), we mean the ring

\[
R_\sigma[x] = \left\{ \sum_{i \geq 0} r_i x^i : r_i \in R \text{ and almost all } r_i \text{ are zero} \right\}
\]

with addition componentwise and multiplication defined by the rule \( xr = \sigma(r)x \). By the skew group ring \( R_\sigma \langle x \rangle \), we mean the ring

\[
R_\sigma \langle x \rangle = \left\{ \sum_{i \in \mathbb{Z}} r_i x^i : r_i \in R \text{ and almost all } r_i \text{ are zero} \right\}
\]
with addition componentwise and multiplication defined by \( x'r = \sigma'(r)x' \).

1.2. An element \( r \in R \) is said to be \( \sigma \)-nilpotent if for all natural integers \( m \) there exists an integer \( n = n(m) > 2 \), depending on \( m \), such that
\[
\sigma^m(r) \cdots \sigma^{m(n-1)}(r) = 0.
\]
It is said to be \( \sigma \)-nilpotent of bounded index if we can find a common integer \( n \) such that the above equality is satisfied for all natural integers \( m \), and the least such \( n \) is called the \( \sigma \)-nilpotency index of \( r \). An ideal \( I \) of \( R \) is said to be \( \sigma \)-nil if \( \sigma(I) = I \) and every element of \( I \) is \( \sigma \)-nilpotent.

2. Main results. In Theorem 3.1 of [1] by Bedi and the author, it is proved that
\[
J(\sigma(I)) = \sum_{i=0}^{\infty} Ix^i + \cdots + Ix^n + \cdots,
\]
where \( I = \{ r \in R : rx \in J(\sigma(I)) \} \). Here \( \sigma(I) = I \). Further, for all \( r \in I \), \( rx^n \in J(\sigma(I)) \), and so \( rx^n \) is nilpotent by [1, Lemma 2.4]. This proves that \( I \) is \( \sigma \)-nil ideal of \( R \). Hence if \( J(\sigma(I)) \neq 0 \) then there exists a nonzero \( \sigma \)-nil ideal. We will use this result without any further mention.

**Theorem 2.1.** Let \( R \) be a ring satisfying the maximal condition on annihilator left ideals and \( \sigma \) be an automorphism of \( R \). Then the following are equivalent:

1. \( J(\sigma(x)) \) is nonzero.
2. \( J(\sigma(x)) \) is nonzero.
3. \( R \) has a nonzero \( \sigma \)-nil ideal.
4. \( R \) has a nonzero right nil ideal.
5. \( P(R) \) is nonzero.
6. \( P(\sigma(x)) \) is nonzero.
7. \( P(\sigma(x)) \) is nonzero.

**Proof.** To prove the theorem first we show that \( (1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (4) \Rightarrow (5) \Rightarrow (6) \Rightarrow (1) \), and then we show that \( (6) \Rightarrow (7) \Rightarrow (2) \).

(1) \Rightarrow (2). This result follows from [1, Theorem 3.1].
(2) \Rightarrow (3). If \( J(\sigma(x)) \) then the ring \( R \) has a nonzero \( \sigma \)-nil ideal.
(3) \Rightarrow (4). Let \( I \) be a nonzero \( \sigma \)-nil ideal of \( R \). Assume \( I \) is not nil and choose \( r \) in \( I \) such that \( r \) is not nilpotent. We claim that there exists \( n \geq 1 \) such that \( rs^n(r) \neq 0 \). Let us suppose that \( rs^n(r) = 0 \) for all natural integers \( n \). Define
\[
I_m = \sigma^m(r)R + \sigma^{m+1}(r)R + \cdots + \cdots
\]
for all natural integers \( m \). Clearly
\[
I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots \supseteq \cdots.
\]
Thus
\[
\text{Ann}_I(I_1) \subseteq \text{Ann}_I(I_2) \subseteq \text{Ann}_I(I_3) \subseteq \cdots \subseteq \cdots.
\]
Let \( \text{Ann}_I(I_1) = \text{Ann}_I(I_{i+1}) \). Since \( rs^n(r) = 0 \) for all natural integers \( n \), \( \sigma'(r) \in \text{Ann}_I(I_{i+1}) = \text{Ann}_I(I_i) \). Thus
\[
\sigma'(r)I_i = 0 \Rightarrow \sigma'(r)\sigma'(r) = 0 \Rightarrow r^2 = 0.
\]
Thus, there exists \( n \geq 1 \) such that \( r\sigma^n(r) \neq 0 \). However \( r\sigma^n(r) \cdots \sigma^m(r) = 0 \) for some \( t > 1 \). We assume that \( t \) is minimal. Put \( s = r\sigma^n(r) \cdots \sigma^{(t-1)n}(r) \). If the right ideal \( sR \) is not nil, \( sk \) is not nilpotent for some \( k \). Put \( r_1 = r \) and \( r_2 = sk \). We show that \( \text{Ann}_j(r_1) \subseteq \text{Ann}_j(r_2) \). Since \( r\sigma^n(r) \cdots \sigma^m(r) = 0 \), \( \sigma^{-n}(r) \in \text{Ann}_j(r_2) \). But \( \sigma^{-n}(r) \notin \text{Ann}_j(r_1) \). Thus \( \text{Ann}_j(r_1) \subseteq \text{Ann}_j(r_2) \). If \( r_2R \) is not nil, then arguing as before we get \( r_3 \) such that \( \text{Ann}_j(r_1) \subseteq \text{Ann}_j(r_2) \subseteq \text{Ann}_j(r_3) \). Continuing in this fashion we will get a nonzero nil right ideal.

(4) \( \Rightarrow \) (5). This follows from [6, Lemma 4.7].

(5) \( \Rightarrow \) (6). Let

\[
\mathfrak{T} = \{ \text{Ann}_j(Rx) | xRx = 0, x \neq 0 \}.
\]

Since \( P(R) \neq 0 \), \( \mathfrak{T} \) is nonempty. Let \( \text{Ann}_j(Rr) \) be a maximal element of \( \mathfrak{T} \). We first show that \( rR\sigma^n(r) = 0 \) for all integers \( n \). If it is not true, then \( r\sigma^m(ar) \neq 0 \) for an integer \( m \). Let \( s = \sigma^{-m}(r)ar \). Clearly \( sRs = 0 \) and \( \text{Ann}_j(Rs) \supseteq \text{Ann}_j(Rr) \). This inclusion is strict because \( \sigma^{-m}(r)Rr \neq 0 \) and

\[
\sigma^{-m}(r)Rs = \sigma^{-m}(r)R\sigma^{-m}(r)ar = 0.
\]

Thus \( rR\sigma^n(r) = 0 \) for all integers \( n \).

Now, observe that \( rR\sigma^n(x) \neq 0 \). Hence \( P(R_\sigma(x)) \) is nonzero.

(6) \( \Rightarrow \) (1). This is clear.

(6) \( \Rightarrow \) (7). Since \( P(R_\sigma(x)) \neq 0 \), let \( I \) be a nonzero nilpotent ideal of \( R_\sigma(x) \). Now, \( I \cap R_\sigma[x] \) is a nonzero nilpotent ideal of \( R_\sigma[x] \). Hence \( P(R_\sigma[x]) \neq 0 \).

(7) \( \Rightarrow \) (2). This is clear.

3. An example. In this section, we give an example of a commutative semisimple algebra \( R \) and an automorphism \( \sigma \) such that \( R_\sigma[x] \) is prime but the Levitzki radical of \( R_\sigma[x] \) is nonzero. In the construction of the example we use the result:

Let \( G(k, m) \) denote the least integer such that if \( g \geq G(k, m) \) and if \( A = \{ a_n \}_{n=0}^{g-1} \) is a strictly increasing sequence of integers with bounded gaps \( a_n - a_{n-1} \leq m \), \( 1 \leq n \leq g-1 \), then \( A \) contains a \( k \)-term arithmetic progression. The number \( G(k, m) \) does exist [5]. The existence of \( G(k, m) \) is an easy consequence of Van der Waerden’s theorem [3, 7].

**Theorem 3.1.** Let \( R \) be a commutative ring and \( \sigma \) be an automorphism of \( R \). If \( r \in R \) is a nonzero \( \sigma \)-nilpotent element of bounded index then \( rxR_\sigma[x] \) is a locally nilpotent right ideal of \( R_\sigma[x] \).

**Proof.** Let \( I = rxR_\sigma[x] \). Clearly \( I \) is a right ideal of \( R_\sigma[x] \). To prove that \( I \) is locally nilpotent, it suffices to prove that the subring \( S \) generated by

\[
T = \{ rs_1x^{i_1}, rs_2x^{i_2}, \ldots, rs_nx^{i_n} \}
\]

is nilpotent for all \( s_j \in R, i_j \geq 1 \). Let \( m = \max(i_1, i_2, \ldots, i_n) \) and \( k \) be the \( \sigma \)-nilpotency index of \( r \). Let \( l = G(k, m) \). We claim that \( S' = 0 \). For this, it suffices to prove that the product of any \( l \) monomials, each of the type given in \( T \), is zero. Any typical
such product $P$ is given by

$$P = \left( rs_{i_1} x^{t_{i_1}} \right) \left( rs_{i_2} x^{t_{i_2}} \right) \cdots \left( rs_{i_l} x^{t_{i_l}} \right)$$

$$= \left( r \sigma^{t_{i_1}}(r) \sigma^{t_{i_2}}(r) \cdots \sigma^{t_{i_l}+t_{i_{l-1}}}(r) \cdots \right)(\cdots) x^{t_{i_1}+t_{i_2}}.$$

Thus, it follows that $P = 0$, since the set

$$\{0, i_{i_1}, i_{i_1} + i_{i_2}, \ldots, i_{i_1} + i_{i_2} + \cdots + i_{i_{l-1}}\}$$

contains $k$ numbers in arithmetic progression, and $k$ is the $\sigma$-nilpotency index of $r$.

Here we raise a question: Let $R$ be a commutative ring. Is $J(R_\sigma[x])$ nonzero if and only if $\mathcal{E}(R_\sigma[x])$ is nonzero? If $J(R_\sigma[x])$ is nonzero then $R$ has a nonzero $\sigma$-nil ideal, say $I$, and hence Theorem 3.1 settles this question in the affirmative, in the case when $I$ contains a $\sigma$-nilpotent element of bounded index.

**Example 3.2.** Let $K$ be an infinite field and $S$ be a polynomial ring over $K$ in the commuting indeterminates $x_i$ indexed by the set of integers $\mathbb{Z}$. Let $\sigma$ be a map from $S$ to $S$ such that

$$\sigma(x_i) = x_{i+1} \quad \text{and} \quad \sigma(k) = k \quad \text{for all } k \in \mathbb{Z}.$$

Then $\sigma$ is an automorphism of $S$. Consider the ideal $I$ of $S$ generated by the set $T = \{x_k x_{k+a} x_{k+2a} : k, a \in \mathbb{Z} \text{ and } a \neq 0\}$. Since $\sigma$ permutes the elements of $T$, $\sigma(I) = I$. Let $R = S/I$. Also, $\sigma$ induces an automorphism of $R$ and we denote this automorphism again by $\sigma$. We can think of $R$ as a $k$-linear span of the monomials

$$\left\{ x_{i_1}^a x_{i_2}^a \cdots x_{i_k}^a : \text{no three elements of } i_1, i_2, \ldots, i_k \end{align*}$$

with usual addition, and we multiply them by using the rule $x_k x_{k+a} x_{k+2a} = 0$ for all $k, a \in \mathbb{Z}$ with $a \neq 0$.

Now, we prove the following: (i) $R$ is semisimple; (ii) $R_\sigma[x]$ is prime; (iii) $\mathcal{E}(R_\sigma[x])$ is nonzero.

(i) $R$ is semisimple. First, we prove that if $J(R) \neq 0$ then $J(R)$ contains a monomial. If $0 \neq r \in J(R)$ then we write $r$ as a polynomial in one of the indeterminates, say $x_{n_1}$; let

$$r = r_0 + r_1 x_{n_1} + \cdots + r_{n_1} x_{n_1}^{n_1}.$$ 

Let $\alpha_1, \alpha_2, \ldots, \alpha_{n_1+1}$ be distinct nonzero elements of $K$ and consider the algebra automorphisms $\theta_i$ of $R$, given by

$$\theta_i(x_{n_1}) = \alpha_i x_{n_1} \quad \text{and} \quad \theta_i(x_j) = x_j \quad \text{if } j \neq n_1.$$

Now $\theta_i(r) = r_0 + r_1 \alpha_1 x_{n_1} + \cdots + r_{n_1} \alpha_1^{n_1} x_{n_1}^{n_1} \in J(R)$ for all $i$. These equations give that $r_{n_1} x_{n_1}^{n_1}$ is a $K$-linear combination of $\theta_1(r), \theta_2(r), \ldots, \theta_{n_1+1}(r)$, and thus, $r_{n_1} x_{n_1}^{n_1} \in J(R)$. Again, we write the element $r_{n_1} x_{n_1}^{n_1}$ as a polynomial in some other indeterminate and repeat the process. Continuing this way, we finally get that $0 \neq s = k x_{n_1}^{n_1} x_{n_2}^{n_2} \cdots x_{n_m}^{n_m} \in J(R)$. Now, since $R$ is a graded ring under the obvious grading and $s$ is a monomial in $J(R)$, by [1, Lemma 2.4], $s$ is nilpotent; however, this is not possible because no three elements of $n_1, n_2, \ldots, n_m$ are in arithmetic progression.
(ii) $R_\sigma[x]$ is prime. Let $f(x) = a_0 + a_1x + \cdots + a_nx^n$ and $g(x) = b_0 + b_1x + \cdots + b_mx^m$ with nonzero $a_n$ and $b_m$ such that

$$f(x)R_\sigma[x]g(x) = 0 \Rightarrow f(x)x^jg(x) = 0 \quad \text{for all } j \geq 0.$$ 
Comparing the coefficient of highest degree term we get that

$$a_n\sigma^{n+j}(b_m) = 0 \quad \text{for all } j \geq 0.$$ 
Thus, if $R_\sigma[x]$ is not prime, there exists $a = a_n$ and $b = b_m$ such that

$$a\sigma^k(b) = 0 \quad \text{for all } k \geq n.$$ 
However, this is not possible and this can be shown by selecting a sufficiently large integer $k$, so that the indeterminates involved in $\sigma^k(b)$ are disjoint from those involved in $a$, and the product of any two monomials in $a\sigma^k(b)$ does not contain three indeterminates whose suffixes are in arithmetic progression.

(iii) $\mathcal{E}(R_\sigma[x])$ is nonzero. An element $x_1 \in R$ is $\sigma$-nilpotent of bounded index ($\sigma$-nilpotency index of $x_1$ is three) and hence by Theorem 3.1, $\mathcal{E}(R_\sigma[x])$ is nonzero.

ACKNOWLEDGEMENTS. The author is thankful to Dr. R. N. Gupta and the referee for useful suggestions.

REFERENCES


Government College, Ajnala, District Amritsar, Panjab, India