Abstract. \( \mathfrak{N}_6^2 \) is a scheme parametrizing pairs \( L \to C \) of smooth algebraic curves \( C \) of genus 10 together with line bundles \( L \) of degree 6 such that \( H^0(C, L) \geq 3 \). It is shown that one of the irreducible components of this scheme is nonreduced at every point.

Introduction. In [ACGH] a general method is outlined for computing the dimension of the tangent space \( T \mathfrak{N}_g^r \). \( \mathfrak{N}_g^r \) is the scheme parametrizing pairs \( L \to C \) in which \( C \) is a smooth curve of genus \( g \), \( L \) is a line bundle such that

1. \( \deg_c(L) = d \),
2. \( \dim H^0(C, L) = h^0(C, L) \geq r + 1 \),

and \( l = L \to C \) is a “generic” point on \( \mathfrak{N}_g^r \). Specifically, \( l \in \mathfrak{N}_g^r \setminus \mathfrak{N}_g^{r+1} \).

The dimension of this tangent space is determined by the kernels of two naturally defined maps on the cohomology of \( C \) in \( L \) and \( K_C \). These maps, \( \mu_0 \) and \( \mu_1 \), together with their kernels will be computed explicitly in the case \( l \in \mathfrak{N}_{6,10}^2 \setminus \mathfrak{N}_{6,10}^{2+1} \) and \( C \) a hyper-elliptic curve. This calculation will then be used to show that \( \mathfrak{N}_{6,10}^2 \) has a nonreduced component.

It will be clear that these computations generalize to higher genus and degree hyper-elliptic curves, but the actual calculations become fairly intractable. For this reason only the case of \( \mathfrak{N}_{6,10}^2 \) will be considered here.

1. We begin with a review of the basic set up and terminology of [ACGH]. For a fixed, smooth curve \( C \) of genus \( g \), define

\[
\text{Pic}^d(C) = \{ L | \deg_c(L) = d \} \\
\cup \\
W^d_c(C) = \{ L | \deg_c(L) = d \text{ and } h^0(C, L) \geq r + 1 \}.
\]

It is well known that

\[
T_L(\text{Pic}^d(C)) \cong H^1(C, \mathcal{O}_C).
\]
A first order variation of a line bundle $L$ on $C$ is a commuting diagram,
\[
\begin{array}{c}
L & \rightarrow & \mathcal{L} \\
\downarrow & & \downarrow \\
C & \rightarrow & C \times S_1 \\
\downarrow & & \downarrow \\
S_0 & \rightarrow & S_1
\end{array}
\]
where $S_n = \text{Spec}(\mathbb{C}[t]/(t^{n+1}))$. The set of isomorphism classes of such diagrams can be canonically identified with $H^1(C, \mathcal{O}_C)$.

The obstruction to extending a section $s \in H^0(C, L)$ to a section of $\mathcal{L}$ can be seen to be $\varphi \cdot s \in H^1(C, L)$, where
\[
\cdot \cdot \cdot H^1(C, \mathcal{O}_C) \otimes H^0(C, L) \rightarrow H^1(C, L)
\]
is the usual cup product, and $\varphi \in H^1(C, \mathcal{O}_C)$ is the element corresponding to $\mathcal{L}$. Thus, one sees that, at a point $L \in W_d^{r+1}(C)$, a tangent vector $\varphi$ in $H^1(C, \mathcal{O}_C) = T_L(\text{Pic}^d(C))$ is also in $T_L(W_d^r(C))$ if and only if $\varphi: H^0(C, L) \rightarrow H^1(C, L)$ is the zero map. In other words, $\varphi$ is in the kernel of the map
\[
H^1(C, \mathcal{O}_C) \rightarrow \text{Hom}(H^0(C, L), H^1(C, L)).
\]
Dualizing yields the map
\[
\mu_0: H^0(C, L) \otimes H^0(C, KL^{-1}) \rightarrow H^0(C, K)
\]
such that
\[
T_L(W_d^r(C)) = \text{Image } \mu_0.
\]
in $h^0(C, L) = r + 1$.

Now the Riemann-Roch theorem implies $h^0(C, KL^{-1}) = g - d + r$. So,
\[
\dim(\text{Im } \mu_0)^\perp = g - \dim(\text{Im } \mu_0) = g - ((r + 1)(g - d + r) - \dim(\ker \mu_0))
\]
\[
= (g - (r + 1)(g - d + r)) + \dim(\ker \mu_0) = \def \rho + \dim(\ker \mu_0).
\]
The number $\rho$ is called the Brill-Noether number. So, at a generic $L$,
\[
\dim T_L(W_d^r(C)) = \rho + \dim(\ker \mu_0).
\]
Next we allow $C$ to vary and define (with $g$ fixed)
\[
\text{Pic}^d = \{ L \rightarrow C | \deg_C(L) = d, g(C) = g \}
\]
\[
\cup
\]
\[
\mathbb{M}_d^r = \{ L \rightarrow C | \deg_C(L) = d, g(C) = g, \text{ and } h^0(C, L) \geq r + 1 \}.
\]
The fact that these are well-defined schemes with good properties is nontrivial, but is studied carefully in [ACGH].

The tangent space $T_L(\text{Pic}^d)$ can be identified with $H^1(C, \Sigma_L)$, where $\Sigma_L$ is the extension of $\mathcal{O}_C$ by $\Theta_C$,
\[
0 \rightarrow \mathcal{O}_C \rightarrow \Sigma_L \rightarrow \Theta_C \rightarrow 0,
\]
SPECIAL LINEAR SYSTEMS ON ALGEBRAIC CURVES

This leads (nontrivially) to the definition of \( \mu_1: \ker(\mu_0) \to H^0(C, K^2) \) given by

\[
\mu_1 \left( \sum_i s_i \otimes r_i \right) = \frac{\partial s_i}{\partial z_a} \otimes r_i \in H^0(C, K^2),
\]

where \( \sum_i s_i \otimes r_i \) is in \( \ker(\mu_0) \).

Again, by studying obstructions, it can be seen that when \( h^0(C, L) = r + 1 \), we have

\[
(*) \quad \dim T_i(\mathcal{O}_6, g) = 3g - 3 + \rho + \dim(\ker(\mu_1)).
\]

Thus in order to compute the dimension of \( T_i(\mathcal{O}_6, g) \), one must first find \( \ker(\mu_0) \) and then the dimension of \( \ker(\mu_1) \).

2. Begin by fixing \( l = L \to C \in \mathcal{O}_6, g - \mathcal{O}_6, g \), where \( C \) is a smooth, genus 10, hyper-elliptic curve. A simple investigation of the map associated to the linear system \( |L|, \varphi_l: C \to \mathbb{P}^2 \) (see \cite{Griffin 2}) shows that, in this case, \( |L| \) has two basepoints and is of the form \( |L| = 2g_2 + Q + R \), with \( Q \neq iR \), where \( i: C \to C \) is the involution on \( C \). We may assume \( |L| = |4P| + Q + R \), where \( P \) is a Weierstrass point on \( C \).

In order to compute \( \mu_0 \), choose a basis for \( H^0(C, L) = H^0(C, 4P + Q + R) \), say \( x_0, x_1, x_2 \), where \( \text{ord}_P(x_i) = 2i \). Then a basis for

\[
H^0(C, KL^{-1}) = H^0(C, \mathcal{O}_C(10P + iQ + iR))
\]

is

\[
\{x_0^2, x_0x_1, x_0x_2, x_1x_2, x_2^2, y_4\} \quad \text{where } \text{ord}_P(y_4) = 10.
\]

On the other hand, \( H^0(C, K) = H^0(C, \mathcal{O}_C(18P)) \) has basis

\[
\{x_0^3, x_0^2x_1, x_0^2x_2, x_0x_1x_2, x_0x_2^2, x_1x_2^2, x_2^3, x_2y_4, z_5, z_6\} \quad \text{where } \text{ord}_P(z_j) = 2j + 6.
\]

From this it is clear that \( \mu_0 \) is neither injective nor surjective (\( \mu_0 \) just "multiplies" \( x_i \otimes x_jx_k \to x_ix_jx_k \)). Its kernel has dimension 10 and basis given in

\textbf{Table 1}

\[
\begin{align*}
\nu_1: & x_0 \otimes x_0x_1 - x_1 \otimes x_0^2 \\
\nu_2: & x_0 \otimes x_0x_2 - x_2 \otimes x_0^2 \\
\nu_3: & x_0 \otimes x_1x_2 - x_1 \otimes x_0x_2 \\
\nu_4: & x_0 \otimes x_1x_2 - x_2 \otimes x_0x_1 \\
\nu_5: & x_0 \otimes x_2^2 - x_2 \otimes x_0x_2 \\
\nu_6: & x_0 \otimes y_4 - x_2 \otimes x_1x_2 \\
\nu_7: & x_1 \otimes x_1x_2 - x_2 \otimes x_0x_2 \\
\nu_8: & x_1 \otimes x_2^2 - x_2 \otimes x_1x_2 \\
\nu_9: & x_1 \otimes y_4 - x_2 \otimes x_1x_2 \\
\nu_{10}: & x_0 \otimes x_0x_2 - x_1 \otimes x_0x_1
\end{align*}
\]
The elements $v_6, v_7, v_9, v_{10}$ come from relations in $H^0(C, K)$, namely $x_0x_2 - x_1^2 = 0, x_0y_4 - x_1x_2^2 = 0$ and $x_1y_4 - x_2^2 = 0$. The other elements are Koszul relations.

In order to compute $\ker(\mu_1)$ we will need

$$\frac{\partial x_{2\alpha}}{\partial z_{\alpha}} = \frac{\partial}{\partial z_{\alpha}} \left( \frac{x_{1\alpha}^2}{x_{0\alpha}} \right) = \frac{2x_{1\alpha}x_{0\alpha}(\partial x_{1\alpha}/\partial z_{\alpha}) - x_{1\alpha}^2(\partial x_{0\alpha}/\partial z_{\alpha})}{x_{0\alpha}^2}$$

which comes from the relation $x_0x_2 - x_1^2$. The $\alpha$ subscript will be dropped (for obvious reasons) and the dependence will be denoted by \{\}'s.

We now give a table of $w_i = \mu_1(v_i)$.

Table 2

<table>
<thead>
<tr>
<th>$w_1$</th>
<th>${x_0x_1(\partial x_0/\partial z) - x_1^2(\partial x_1/\partial z)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$w_2$</td>
<td>${2x_0x_2(\partial x_0/\partial z) - 2x_0x_1(\partial x_1/\partial z)}$</td>
</tr>
<tr>
<td>$w_3$</td>
<td>${x_1x_2(\partial x_0/\partial z) - x_0x_2(\partial x_1/\partial z)}$</td>
</tr>
<tr>
<td>$w_4$</td>
<td>${2x_1x_2(\partial x_0/\partial z) - 2x_0x_2(\partial x_1/\partial z)}$</td>
</tr>
<tr>
<td>$w_5$</td>
<td>${2x_2^2(\partial x_0/\partial z) - 2x_1x_2(\partial x_1/\partial z)}$</td>
</tr>
<tr>
<td>$w_6$</td>
<td>${2y_4(\partial x_0/\partial z) - 2x_2^2(\partial x_1/\partial z)}$</td>
</tr>
<tr>
<td>$w_7$</td>
<td>${x_2^2(\partial x_0/\partial z) - x_1x_2(\partial x_1/\partial z)}$</td>
</tr>
<tr>
<td>$w_8$</td>
<td>${y_4(\partial x_0/\partial z) - x_2^2(\partial x_1/\partial z)}$</td>
</tr>
<tr>
<td>$w_9$</td>
<td>${(x_1y_4/x_0)(\partial x_0/\partial z) - y_4(\partial x_1/\partial z)}$</td>
</tr>
<tr>
<td>$w_{10}$</td>
<td>${x_0x_2(\partial x_0/\partial z) - x_0x_1(\partial x_1/\partial z)}$</td>
</tr>
</tbody>
</table>

Trivially, $w_2 = 2w_{10}, w_4 = 2w_3, w_5 = 2w_7, w_6 = 2w_8$, and in $H^0(C, K^2L)$ we have

$$x_0w_2 = 2x_1w_1, \quad x_1w_2 = 2x_0w_3, \quad x_0w_3 = 2x_1w_3, \quad x_0w_6 = x_1w_5 \quad \text{and} \quad x_1w_6 = 2x_0w_9.$$  

This clearly implies there can be no linear relation among $\{w_1, w_2, w_3, w_5, w_6, w_9\}$ in $H^0(C, K^2)$. Consequently, the kernel of $\mu_1$ has basis $\{w_2 - 2w_{10}, w_4 - 2w_3, w_5 - 2w_7, w_6 - 2w_8\}$. By equation (*) we finally have

$$\dim T_l(\mathcal{O}^2_{6,10}) = 3g - 3 + \rho + \dim(\ker\mu_1)$$

$$= 3(10) - 3 + 10 - 3(10 - 6 + 2) + 4 = 23.$$  

It is easy to compute the dimension of the hyper-elliptic component $W$ of $\mathcal{O}^2_{6,10}$. Since the $g_2$ on a hyper-elliptic curve is unique and $l = L \rightarrow C$, where $|L| = 2g_2 + P + Q$, one has immediately that

$$\dim W = \dim\{\text{genus 10, hyper-elliptic curves}\} + 2.$$  

By Hurwitz’s formula,

$$\dim\{\text{hyper-elliptic curves of genus } g\} = 2g - 1.$$  

Therefore,

$$\dim W = 19 + 2 = 21!$$
Thus, since $\dim T_l(\mathcal{W}_{6,10}^2) = 23 > \dim T_l(\mathcal{W}_{6,10}^2) = 21$, we conclude

**Theorem.** The “component of hyper-elliptics” in $\mathcal{W}_{6,10}^2$ is nonreduced.

A final remark: In general the singularities of $\mathcal{W}_{d,g}$ are “worst” along $\mathcal{W}_{d,g}^{r+1}$, and in fact one can show, using these methods, that if $l \in \mathcal{W}_{6,10}^3$, 

$$\dim T_l(\mathcal{W}_{6,10}^2) > 23.$$ 

In spite of this, $\mathcal{W}_{6,10}^3$ is smooth, i.e.

$$\dim T_l(\mathcal{W}_{6,10}^3) = \dim T_l(\mathcal{W}_{6,10}^3) = 19$$

for all $l \in \mathcal{W}_{6,10}^3$. So it is very important to distinguish between $T_l(\mathcal{W}_{d,g}^{r+1})$ and $T_l(\mathcal{W}_{d,g}^r)$ when $l \in \mathcal{W}_{d,g}^{r+1} \subset \mathcal{W}_{d,g}^r$.

**REFERENCES**


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