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ON A FAMILY OF SPECIAL LINEAR SYSTEMS ON ALGEBRAIC CURVES[1]

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ABSTRACT. $\mathfrak{W}_6^2$ is a scheme parametrizing pairs $L \to C$ of smooth algebraic curves C of genus 10 together with line bundles L of degree 6 such that $H^0(C, L) \geqslant 3$. It is shown that one of the irreducible components of this scheme is nonreduced at every point.

Introduction. In [**ACGH**] a general method is outlined for computing the dimension of the tangent space $T_l(\mathfrak{W}_{d,g}^r)$. $\mathfrak{W}_{d,g}^r$ is the scheme parametrizing pairs $L \to C$ in which C is a smooth curve of genus g, L is a line bundle such that

(1) $\deg_C(L) = d$,

(2) $\dim H^0(C, L) = h^0(C, L) \geqslant r + 1$,

and $l = L \to C$ is a "generic" point on $\mathfrak{W}_{d,g}^r$. Specifically, $l \in \mathfrak{W}_{d,g}^r - \mathfrak{W}_{d,g}^{r+1}$.

The dimension of this tangent space is determined by the kernels of two naturally defined maps on the cohomology of C in L and K_C. These maps, μ_0 and μ_1, together with their kernels will be computed explicitly in the case $l \in \mathfrak{W}_{6,10}^2 - \mathfrak{W}_{6,10}^3$ and C a hyper-elliptic curve. This calculation will then be used to show that $\mathfrak{W}_{6,10}^2$ has a *nonreduced* component.

It will be clear that these computations generalize to higher genus and degree hyper-elliptic curves, but the actual calculations become fairly intractable. For this reason only the case of $\mathfrak{W}_{6,10}^2$ will be considered here.

1. We begin with a review of the basic set up and terminology of [**ACGH**]. For a fixed, smooth curve C of genus g, define

$$\begin{aligned} &\mathrm{Pic}^d(C) = \{L \mid \deg_C(L) = d\} \\ &\cup\mid \\ &W_d^r(C) = \{L \mid \deg_C(L) = d \text{ and } h^0(C, L) \geqslant r + 1\}. \end{aligned}$$

It is well known that

$$T_L(\mathrm{Pic}^d(C)) \simeq H^1(C, \mathcal{O}_C).$$

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A *first order variation* of a line bundle L on C is a commuting diagram,

$$\begin{array}{ccc} L & \to & \mathfrak{L} \\ \downarrow & & \downarrow \\ C & \to & C \times S_1 \\ \downarrow & & \downarrow \\ S_0 & \hookrightarrow & S_1 \end{array}$$

where $S_n = \operatorname{Spec}(\mathbf{C}[t]/(t^{n+1}))$. The set of isomorphism classes of such diagrams can be canonically identified with $H^1(C, \mathcal{O}_C)$.

The obstruction to extending a section $s \in H^0(C, L)$ to a section of $\mathfrak{L}$ can be seen to be $\varphi \cdot s \in H^1(C, L)$, where

$$\text{“}\cdot\text{”}\ H^1(C, \mathcal{O}_C) \otimes H^0(C, L) \to H^1(C, L)$$

is the usual cup product, and $\varphi \in H^1(C, \mathcal{O}_C)$ is the element corresponding to $\mathfrak{L}$. Thus, one sees that, at a point $L \in W_d^r(C) - W_d^{r+1}(C)$, a tangent vector φ in $H^1(C, \mathcal{O}_C) = T_L(\operatorname{Pic}^d(C))$ is also in $T_L(W_d^r(C))$ if and only if φ: $H^0(C, L) \to H^1(C, L)$ is the zero map. In other words, φ is in the kernel of the map

$$H^1(C, \mathcal{O}_C) \to \operatorname{Hom}(H^0(C, L), H^1(C, L)).$$

Dualizing yields the map

$$\mu_0 \colon H^0(C, L) \otimes H^0(C, KL^{-1}) \to H^0(C, K)$$

such that

$$T_L(W_d^r(C)) = (\operatorname{Image} \mu_0)^{\perp}$$

in $h^0(C, L) = r + 1$.

Now the Riemann-Roch theorem implies $h^0(C, KL^{-1}) = g - d + r$. So,

$$\begin{aligned} \dim(\operatorname{Im} \mu_0)^{\perp} &= g - \dim(\operatorname{Im} \mu_0) = g - ((r+1)(g-d+r) - \dim(\ker \mu_0)) \\ &= (g - (r+1)(g-d+r)) + \dim(\ker \mu_0) =_{\text{def}} \rho + \dim(\ker \mu_0). \end{aligned}$$

The number ρ is called the Brill-Noether number. So, at a generic L,

$$\dim T_L(W_d^r(C)) = \rho + \dim(\ker \mu_0).$$

Next we allow C to vary and define (with g *fixed*)

$$\begin{aligned} &\operatorname{Pic}^d = \{L \to C \mid \deg_C(L) = d, g(C) = g\} \\ &\cup\mid \\ &\mathcal{W}_d^r = \{L \to C \mid \deg_C(L) = d, g(C) = g, \text{and } h^0(C, L) \geqslant r + 1\}. \end{aligned}$$

The fact that these are well-defined schemes with good properties is nontrivial, but is studied carefully in **[ACGH]**.

The tangent space $T_l(\operatorname{Pic}^d)$ can be identified with $H^1(C, \Sigma_L)$, where Σ_L is the extension of $\mathcal{O}_C$ by Θ_C,

$$0 \to \mathcal{O}_C \to \Sigma_L \to \Theta_C \to 0,$$

with extension class

$$\omega = c_1(L) \in H^1\big(C, \underline{\mathrm{Hom}}(\Theta, \mathcal{O})\big) \simeq H^1(C, K).$$

This leads (nontrivially) to the definition of μ_1: $\ker(\mu_0) \to H^0(C, K^2)$ given by

$$\mu_1\Big(\sum_i s_i \otimes r_i\Big) = \frac{\partial s_{i\alpha}}{\partial z_\alpha} \otimes r_{i\alpha} \in H^0(C, K^2),$$

where $\Sigma_i s_i \otimes r_i$ is in $\ker(\mu_0)$.

Again, by studying obstructions, it can be seen that when $h^0(C, L) = r + 1$, we have

$$(*) \qquad \dim T_l\big(\mathcal{W}^r_{d,g}\big) = 3g - 3 + \rho + \dim(\ker \mu_1).$$

Thus in order to compute the dimension of $T_l(\mathcal{W}^2_{6,10})$, one must first find $\ker(\mu_0)$ and then the dimension of $\ker(\mu_1)$.

2. Begin by fixing $l = L \to C \in \mathcal{W}^2_{6,10} - \mathcal{W}^3_{6,10}$, where C is a smooth, genus 10, hyper-elliptic curve. A simple investigation of the map associated to the linear system $|L|$, $\varphi_{|L|}$: $C \to \mathbf{P}^2$ (see **[Griffin 2]**) shows that, in this case, $|L|$ has two basepoints and is of the form $|L| = 2g^1_2 + Q + R$, with $Q \neq iR$, where i: $C \to C$ is the involution on C. We may assume $|L| = |4P| + Q + R$, where P is a Weierstrass point on C.

In order to compute μ_0, choose a basis for $H^0(C, L) = H^0(C, 4P + Q + R)$, say x_0, x_1, x_2, where $\mathrm{ord}_P(x_i) = 2i$. Then a basis for

$$H^0(C, KL^{-1}) = H^0(C, \mathcal{O}_C(10P + iQ + iR))$$

is

$$\{x_0^2, x_0x_1, x_0x_2, x_1x_2, x_2^2, y_4\} \quad \text{where } \mathrm{ord}_P(y_4) = 10.$$

On the other hand, $H^0(C, K) = H^0(C, \mathcal{O}_C(18P))$ has basis

$$\{x_0^3, x_0^2x_1, x_0^2x_2, x_0x_1x_2, x_0x_2^2, x_1x_2^2, x_2^3, x_2y_4, z_5, z_6\} \quad \text{where } \mathrm{ord}_P(z_j) = 2j + 6.$$

From this it is clear that μ_0 is neither injective nor surjective (μ_0 just "multiplies" $x_i \otimes x_jx_k \to x_ix_jx_k$). Its kernel has dimension 10 and basis given in

TABLE 1

$$\begin{aligned}
v_1&: x_0 \otimes x_0x_1 - x_1 \otimes x_0^2\\
v_2&: x_0 \otimes x_0x_2 - x_2 \otimes x_0^2\\
v_3&: x_0 \otimes x_1x_2 - x_1 \otimes x_0x_2\\
v_4&: x_0 \otimes x_1x_2 - x_2 \otimes x_0x_1\\
v_5&: x_0 \otimes x_2^2 - x_2 \otimes x_0x_2\\
v_6&: x_0 \otimes y_4 - x_2 \otimes x_1x_2\\
v_7&: x_1 \otimes x_1x_2 - x_2 \otimes x_0x_2\\
v_8&: x_1 \otimes x_2^2 - x_2 \otimes x_1x_2\\
v_9&: x_1 \otimes y_4 - x_2 \otimes x_2^2\\
v_{10}&: x_0 \otimes x_0x_2 - x_1 \otimes x_0x_1
\end{aligned}$$

The elements v_6, v_7, v_9, v_{10} come from relations in $H^0(C, K)$, namely $x_0x_2 - x_1^2 = 0$, $x_0y_4 - x_1x_2^2 = 0$ and $x_1y_4 - x_2^2 = 0$. The other elements are Koszul relations.

In order to compute $\ker(\mu_1)$ we will need

$$\frac{\partial x_{2\alpha}}{\partial z_\alpha} = \frac{\partial}{\partial z_\alpha}\left(\frac{x_{1\alpha}^2}{x_{0\alpha}}\right) = \frac{2x_{1\alpha}x_{0\alpha}(\partial x_{1\alpha}/\partial z_\alpha) - x_{1\alpha}^2(\partial x_{0\alpha}/\partial z_\alpha)}{x_{0\alpha}^2}$$
$$= \frac{2x_{1\alpha}(\partial x_{1\alpha}/\partial z_\alpha) - x_{2\alpha}(\partial x_{0\alpha}/\partial z_\alpha)}{x_{0\alpha}},$$

which comes from the relation $x_0x_2 - x_1^2$. The α subscript will be dropped (for obvious reasons) and the dependence will be denoted by { }'s.

We now give a table of $w_i = \mu_1(v_i)$.

TABLE 2

$$\begin{aligned}
w_1&: \{x_0x_1(\partial x_0/\partial z) - x_0^2(\partial x_1/\partial z)\}\\
w_2&: \{2x_0x_2(\partial x_0/\partial z) - 2x_0x_1(\partial x_1/\partial z)\}\\
w_3&: \{x_1x_2(\partial x_0/\partial z) - x_0x_2(\partial x_1/\partial z)\}\\
w_4&: \{2x_1x_2(\partial x_0/\partial z) - 2x_0x_2(\partial x_1/\partial z)\}\\
w_5&: \{2x_2^2(\partial x_0/\partial z) - 2x_1x_2(\partial x_1/\partial z)\}\\
w_6&: \{2y_4(\partial x_0/\partial z) - 2x_2^2(\partial x_1/\partial z)\}\\
w_7&: \{x_2^2(\partial x_0/\partial z) - x_1x_2(\partial x_1/\partial z)\}\\
w_8&: \{y_4(\partial x_0/\partial z) - x_2^2(\partial x_1/\partial z)\}\\
w_9&: \{(x_1y_4/x_0)(\partial x_0/\partial z) - y_4(\partial x_1/\partial z)\}\\
w_{10}&: \{x_0x_2(\partial x_0/\partial z) - x_0x_1(\partial x_1/\partial z)\}
\end{aligned}$$

Trivially, $w_2 = 2w_{10}$, $w_4 = 2w_3$, $w_5 = 2w_7$, $w_6 = 2w_8$, and in $H^0(C, K^2L)$ we have

$$x_0w_2 = 2x_1w_1, \quad x_1w_2 = 2x_0w_3, \quad x_0w_5 = 2x_1w_3,$$
$$x_0w_6 = x_1w_5 \quad \text{and} \quad x_1w_6 = 2x_0w_9.$$

This clearly implies there can be *no* linear relation among $\{w_1, w_2, w_3, w_5, w_6, w_9\}$ in $H^0(C, K^2)$. Consequently, the kernel of μ_1 has basis $\{w_2 - 2w_{10}, w_4 - 2w_3, w_5 - 2w_7, w_6 - 2w_8\}$. By equation $(*)$ we finally have

$$\begin{aligned}\dim T_l\left(\mathfrak{W}_{6,10}^2\right) &= 3g - 3 + \rho + \dim(\ker \mu_1)\\ &= 3(10) - 3 + 10 - 3(10 - 6 + 2) + 4 = 23.\end{aligned}$$

It is easy to compute the dimension of the hyper-elliptic component W of $\mathfrak{W}_{6,10}^2$. Since the g_2^1 on a hyper-elliptic curve is unique and $l = L \to C$, where $|L| = 2g_2^1 + P + Q$, one has immediately that

$$\dim W = \dim\{\text{genus } 10, \text{hyper-elliptic curves}\} + 2.$$

By Hurwitz's formula,

$$\dim\{\text{hyper-elliptic curves of genus } g\} = 2g - 1.$$

Therefore,

$$\dim W = 19 + 2 = 21!$$

Thus, since $\dim T_l(\mathcal{W}^2_{6,10}) = 23 > \dim_l(\mathcal{W}^2_{6,10}) = 21$, we conclude

THEOREM. *The "component of hyper-elliptics" in* $\mathcal{W}^2_{6,10}$ *is nonreduced.*

A final remark: In general the singularities of $\mathcal{W}^r_{d,g}$ are "worst" along $\mathcal{W}^{r+1}_{d,g}$, and in fact one can show, using these methods, that if $l \in \mathcal{W}^3_{6,10}$,

$$\dim T_l\left(\mathcal{W}^2_{6,10}\right) > 23.$$

In spite of this, $\mathcal{W}^3_{6,10}$ is *smooth*, i.e.

$$\dim T_l\left(\mathcal{W}^3_{6,10}\right) = \dim_l\left(\mathcal{W}^3_{6,10}\right) = 19$$

for all $l \in \mathcal{W}^3_{6,10}$. So it is very important to distinguish between $T_l(\mathcal{W}^{r+1}_{d,g})$ and $T_l(\mathcal{W}^r_{d,g})$ when $l \in \mathcal{W}^{r+1}_{d,g} \subseteq \mathcal{W}^r_{d,g}$.

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