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# TWO HELLY TYPE THEOREMS

KRZYSZTOF KOŁODZIEJCZYK

ABSTRACT. Two Helly type theorems for convex sets containing $k$-dimensional half-flats are established, one of them being an extension of Katchalski's theorem [3] where the case $k = 1$ was considered.

**1. Introduction.** For any set $C$ in linear space let $\operatorname{conv} C$ denote the convex hull of $C$, $\operatorname{aff} C$ the affine hull of $C$, and $\dim C$ the dimension of $\operatorname{aff} C$. By a $k$-flat ($k$-half-flat) we mean a translate of a subspace (half-space) of dimension $k$.

In this note we prove the following two Helly type theorems.

THEOREM 1. *If $\mathfrak{F}$ is a finite family of convex sets in $\mathbf{R}^n$ such that the intersection of any $2n - 2k + 2$ members of $\mathfrak{F}$ contains a $k$-half-flat, then $\bigcap \mathfrak{F}$ contains a $k$-half-flat.*

THEOREM 2. *If $\mathfrak{F}$ is a finite family of convex sets in a linear space such that*:

(1) $\max\{\dim C \mid C \in \mathfrak{F}\} = d$,

(2) $\dim \bigcup \mathfrak{F} = n$,

(3) $\operatorname{card} \mathfrak{F} \geqslant r(n, d, k)$,

*where*

$$r(n, d, k) = \begin{cases} 2d - 2k + 2 & \text{if } 1 \leqslant k \leqslant d, n = d, \\ 2d - 2k + 1 & \text{if } 1 \leqslant k < d, n > d, \\ 2 & \text{if } k = d, n > d, \end{cases}$$

*then $\bigcap \mathfrak{F}$ contains a $k$-half-flat provided the intersection of any $r(n, d, k)$ members of $\mathfrak{F}$ contains a $k$-half-flat.*

The case $k = 1$ in Theorem 1 was proved by Katchalski [3]. Our proof of Theorem 1 employs Katchalski's result and the following theorem of de Santis [2].

*If $\mathfrak{F}$ is a finite family of convex sets in $\mathbf{R}^n$ such that the intersection of any $n - k + 1$ members of $\mathfrak{F}$ contains a $k$-flat, then $\bigcap \mathfrak{F}$ contains a $k$-flat.*

Theorem 2 is analogous to the results by Netrebin [4, Theorems 1 and 2].

Related Helly type theorems can be found in [1].

**2. Proof of Theorem 1.** The validity of the theorem for $k = 1$ follows by Katchalski's result.

Let $1 < k \leqslant n$. Since $2n - 2k + 2 \geqslant n - (k - 1) + 1$, by our assumption and the theorem of de Santis there exists a $(k - 1)$-flat $\mathbf{H}$ such that

$$\mathbf{H} \subset \bigcap \mathfrak{F}.$$

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Now let $\mathbf{H}^*$ be an $(n-k+1)$-flat which is complementary to $\mathbf{H}$ and let

$$\mathfrak{F}^* = \{C^* = C \cap \mathbf{H}^* \mid C \in \mathfrak{F}\}$$

be a family of convex sets in $\mathbf{H}^*$. Choose any subfamily $\mathfrak{F}_1$, $\mathfrak{F}_1 \subset \mathfrak{F}$, consisting of $2n-2k+2$ elements. By assumption there exists a $k$-half-flat $\mathbf{E}$ such that $\mathbf{E} \subset \bigcap \mathfrak{F}_1$. It is clear that $\mathbf{E} \cap \mathbf{H}^*$ contains a 1-half-flat (ray). This shows that any subfamily $\mathfrak{F}_1^*$, $\mathfrak{F}_1^* \subset \mathfrak{F}^*$, consisting of $2n-2k+2$ elements, contains a ray in its intersection, and we can apply Katchalski's theorem to the family $\mathfrak{F}^*$ in $\mathbf{H}^*$. This way we infer that there exists a ray $\lambda$ which is contained in all members of $\mathfrak{F}^*$ and consequently in all members of $\mathfrak{F}$. Without loss of generality we can suppose that an apex of $\lambda$ belongs to $\mathbf{H}$. Now obviously the set $\operatorname{conv}(\mathbf{H} \cup \lambda)$ is a $k$-half-flat and is contained in $\bigcap \mathfrak{F}$. This completes the proof.

**3. Proof of Theorem 2.** In the case of $1 \leqslant k \leqslant d$, $n = d$, Theorem 2 coincides with Theorem 1. The other two cases we prove by induction on card $\mathfrak{F}$. Let $s = \operatorname{card} \mathfrak{F}$ and $r = r(n, d, k)$. The theorem is obviously true for $s = r$. Suppose the result holds for $s = s_0 \geqslant r$ and take $s = s_0 + 1$. Putting

$$B_m = \bigcap \{C_i \mid i \neq m\}, \qquad m = 1, 2, \ldots, s_0 + 1,$$

it can be shown, similarly as in the proof of Theorem 2 in [**4**], that in both cases considered

$$\dim \bigcup \{B_m \mid m = 1, 2, \ldots, s_0 + 1\} = d.$$

Now let

$$\mathbf{B} = \operatorname{aff}\Big( \bigcup \{B_m \mid m = 1, 2, \ldots, s_0 + 1\}\Big).$$

*Case* 1. $1 \leqslant k < d$, $n > d$. Since $n > d$ there is a set, say $C_1$, contained in $\mathfrak{F}$ such that $\dim(C_1 \cap \mathbf{B}) \leqslant d - 1$. Consider the family $\mathfrak{F}' = \{C' = \mathbf{P} \cap C \mid C \in \mathfrak{F}\}$ of convex sets in $\mathbf{P} = \operatorname{aff}(C_1 \cap \mathbf{B})$. Choose any subfamily $\mathfrak{F}_1'$, $\mathfrak{F}_1' \subset \mathfrak{F}'$, containing $2d - 2k$ sets. The following inclusions are obvious:

$$\bigcap \mathfrak{F}_1' = \bigcap \mathfrak{F}_1 \cap \mathbf{P} \supset B_{m_0} \cap \mathbf{P} \supset B_{m_0}$$

for some $m_0$, $2 \leqslant m_0 \leqslant s_0 + 1$. Our induction assumption implies that the set $B_{m_0}$ contains a $k$-half-flat. Hence, by the above inclusions, any subfamily $\mathfrak{F}_1'$ consisting of $2d - 2k$ sets contains a $k$-half-flat in its intersection. This shows that the family $\mathfrak{F}'$ satisfies the conditions of Theorem 1 in the $(d-1)$-flat $\mathbf{P}$ and hence there is a $k$-half-flat which is contained in $\bigcap \mathfrak{F}'$ and consequently in $\bigcap \mathfrak{F}$.

*Case* 2. $k = d$, $n > d$. Consider the family $\mathfrak{F}'' = \{C'' = \mathbf{B} \cap C \mid C \in \mathfrak{F}\}$. Choose any two sets of family $\mathfrak{F}''$ (e.g. $C_1''$ and $C_2''$). Then we have

$$C_1'' \cap C_2'' = \mathbf{B} \cap (C_1 \cap C_2) \supset \mathbf{B} \cap B_{m_0} \supset B_{m_0}$$

for some $m_0$, $3 \leqslant m_0 \leqslant s_0 + 1$. This shows, by our induction hypothesis, that the family $\mathfrak{F}''$ satisfies the conditions of Theorem 1 in the $d$-flat $\mathbf{B}$. Now applying Theorem 1 we complete the proof of Theorem 2.

REMARK. A simple modification of the example given in the last part of the proof of Theorem 2 in [**4**] shows that the number $r(n, d, k)$ cannot be replaced by a smaller number.

## References

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Institute of Mathematics, Technical University of Wrocław, Wybrzeze Wyspiańskiego 27, 50-370 Wrocław, Poland