AN INFINITE-DIMENSIONAL PRE-HILBERT SPACE NOT HOMEOMORPHIC TO ITS OWN SQUARE

ROMAN POL

ABSTRACT. Given an arbitrary infinite-dimensional separable complete linear metric space $X$, there exists a direct sum decomposition $X = V_0 \oplus V_1$ such that each summand $V_i$ intersects every linearly independent Cantor set in $X$ (this decomposition can be considered as a linear analogue to the classical Bernstein's decomposition into totally imperfect sets).

THEOREM. Each summand $V$ of such a decomposition is not homeomorphic to its own square, and if $T : V \rightarrow V$ is a linear bounded operator, then either the kernel or the range of $T$ is finite-dimensional.

In the case of $X = l_2$ this provides an example of a space $V$ with the properties stated in the title, which answers a well-known question, cf. Arhangelskiï [A, Problem 21] and Geoghegan [G, Problem (LS 12)].

1. Bernstein-like direct sum decompositions of infinite-dimensional complete linear separable metric spaces. Let $X$ be an infinite-dimensional separable complete linear metric space (cf. [B-P] for the terminology). In this note we shall consider direct sum decompositions

$$(1) \quad X = V_0 \oplus V_1$$

such that

$$(2) \quad \text{if } C \text{ is a linearly independent Cantor set in } X, \text{ then } V_0 \cap C \neq \emptyset \neq V_1 \cap C;$$

here "Cantor set" means "topologically the Cantor set" and we shall call "Bernstein-like direct sum decomposition of $X" any decomposition of $X$ satisfying (1) and (2).

To obtain a Bernstein-like direct sum decomposition of $X$ one can follow a classical Bernstein's reasoning [Kt, I, §36, proof of Theorem 1]:

Let $\{C_\alpha : \alpha < 2^\omega\}$ be all linearly independent Cantor sets in $X$, and let us choose inductively points $x_\alpha, y_\alpha \in C_\alpha$ such that at each stage $\beta$ of the construction the vectors $\{x_\alpha : \alpha < \beta\} \cup \{y_\alpha : \alpha < \beta\}$ are linearly independent; then let us add to the collection $\{x_\alpha : \alpha < 2^\omega\} \cup \{y_\alpha : \alpha < 2^\omega\}$ a set $A$ to obtain a linear basis in $X$; and finally, let $V_0$ be the linear span of the set $\{x_\alpha : \alpha < 2^\omega\} \cup A$ and let $V_1$ be the linear span of the set $\{y_\alpha : \alpha < 2^\omega\}$.
2. The results.

2.1. **Theorem.** Let $X$ be an infinite-dimensional separable complete linear metric space and let $X = V_0 \oplus V_1$ be a Bernstein-like direct sum decomposition of $X$ (see §1). Then each summand $V$ of the decomposition has the following properties:

(i) $V$ is of second category in $X$;

(ii) $V$ is not homeomorphic to its own square $V \times V$, and moreover,

(iii) if $V$ is homeomorphic to the product $E \times F$ of arbitrary spaces then either $E$ or $F$ contains a nonempty open finite-dimensional subspace;

(iv) if $T: V \to V$ is a linear bounded operator then either the kernel or the range of $T$ is finite-dimensional.

2.2. **Remark.** If we take $X = l_2$, then the space $V$ described in Theorem 2.1 is a pre-Hilbert space not homeomorphic to its own square, which answers a well-known question, cf. Arhangelskii [A, Problem 21] and Geoghegan [G, Problem (LS 12)]. Property (iv) shows also that in the pre-Hilbert space $V$ the only complementable linear subspaces are that of finite dimension or codimension.

3. A theorem of Mycielski on independent Cantor sets. The proof of Theorem 2.1 will be based on the following fact, being a fairly simple consequence of a general theorem on independent sets due to Mycielski [M], cf., also [M2] and [K2].

3.1. **Lemma.** Let $A$ be an analytic set in a separable complete linear metric space $X$. If $A$ contains an uncountable linearly independent set $L$, then $A$ contains a linearly independent Cantor set.

**Proof.** Let $f: P \to A$ be a continuous map of the irrationals $P$ onto $A$, let $M$ be a subset of $P$ which is mapped by $f$ in a one-to-one way onto $L$ and let $Z$ be the set of all condensation points of $M$ in $P$ [K, §18]. Let $R_n = \{(x_1, \ldots, x_n) \in Z^n: \text{the vectors } f(x_1), \ldots, f(x_n) \text{ are linearly dependent}\}$. Then $Z$ is a completely metrizable space and each $R_n$ is closed in $Z^n$. Moreover, $M \cap Z$ is dense in $Z$ and if $x_1, \ldots, x_n$ are distinct points from $M$, then $(x_1, \ldots, x_n) \notin R_n$, so $R_n$ is nowhere dense in $Z^n$. Thus a theorem of Mycielski [M1, Theorem 1] and [M2, Theorem 1] guarantees an existence of a Cantor set $C \subset Z$ such that if $x_1, \ldots, x_n$ are distinct points from $C$ then $(x_1, \ldots, x_n) \notin R_n$, i.e. the vectors $f(x_1), \ldots, f(x_n)$ are linearly independent. It follows that $f$ maps $C$ homeomorphically onto a linearly independent Cantor set $f(C) \subset A$.

4. An auxiliary proposition. The following, rather special proposition, states a property of the space $V$ which is most essential in the proof of Theorem 2.1(iii) in §5.2 (cf. also Remark 7).

4.1. **Proposition.** Let $X = V_0 \oplus V_1$ be a Bernstein-like direct sum decomposition as in Theorem 2.1 and let $V$ be any summand of this decomposition. Assume that $G$ is a $G_\delta$-set in $X$ containing $V$ and that $f: G \to Y$ is a continuous map onto a metrizable space $Y$. Then, either the image $f(V)$ is open modulo a first category set in $Y$, or there exists a point $y \in f(V)$ such that the fiber $f^{-1}(y)$ contains a relatively open nonempty finite-dimensional subset.
Proof. At first, in (I) and (II), we shall recall some facts about the Effros Borel structure (the reader can consult Dellacherie [D, §2] or Christensen [Ch] for this topic); the essential part of the proof is the third one.

(I) Let $F(G)$ be the set of all closed subsets of $G$ and let $B(G)$ be the Effros Borel structure in $F(G)$, i.e. the $\sigma$-algebra generated by the sets

$$U = \{ F \in F(G) : F \cap U \neq \emptyset \}, \quad U \text{ being open in } G.$$

Let $Z$ be a compact metrizable extension of $X$. Then the Effros Borel structure $B(Z)$ in the hyperspace $F(Z)$ of $Z$ coincides with the $\sigma$-algebra of Borel sets of the topological space $F(Z)$ endowed with the exponential topology and the map $F \to \overline{F}$ from $F(G)$ to $F(Z)$ (the closure being taken in $Z$) is $(B(G), B(Z))$-measurable. Since the set of all finite-dimensional compact subsets of $Z$ is in $B(Z)$ [K1, Volume 2, Theorem 4, p. 108], and since this is also true for the set of all compact subsets of $Z$ contained in $X$, the measurability of the map $F \to \overline{F}$ guarantees that

$$E = \{ F \in F(G) : \text{the closure in } X \text{ of } F \text{ is compact and finite-dimensional} \} \in B(G).$$

(II) Given an open set $W$ in $G$, let us define a map $\Phi : Y \to F(G)$ by the formula

$$\Phi(y) = f^{-1}(y) \cap W.$$

Let $A(Y)$ be the $\sigma$-algebra in $Y$ generated by the family of analytic sets; recall, that by a classical theorem of Szpilrajn-Marczewski [K1, II, §35, Corollary 1], each set in $A(Y)$ is open modulo a first category set in $Y$. Let us verify that

$$\Phi \text{ is } (A(Y), B(G))\text{-measurable.}$$

Indeed, if $U$ is a generator of the $\sigma$-algebra $B(G)$ described in (3), then $\Phi^{-1}(U) = \{ y \in Y : f^{-1}(y) \cap W \cap U \neq \emptyset \} = f(U \cap W)$, which is an analytic set.

(III) Let $W_1, W_2, \ldots$ be a countable base in $G$ and let, for each $i$,

$$\Phi_i(y) = f^{-1}(y) \cap W_i.$$

It follows from (4) and (5) that

$$E_i = \Phi_i^{-1}(E \setminus \{ \emptyset \}) \in A(Y),$$

so in particular, each $E_i$ is open modulo a first category set in $Y$. Let us check that

$$Y \setminus f(V) \subset \bigcup_i E_i.$$

If $y \in Y \setminus f(V)$, then $f^{-1}(y)$ is a $G_\delta$-set in $X$ disjoint from $V$ and hence, by Lemma 3.1 and property (2) of $V$, the linear space $L$ spanned by the set $f^{-1}(y)$ has the countable linear dimension. Thus $L$ is a union of countably many finite-dimensional linear subspaces, and therefore $f^{-1}(y)$ can be covered by countably many compact finite-dimensional sets $L_1, L_2, \ldots$. By the Baire category theorem, one of the sets $L_k \cap f^{-1}(y)$ has the nonempty interior in the space $f^{-1}(y)$, and therefore there exists an element $W_i$ in our base such that $\emptyset \neq f^{-1}(y) \cap W_i \subset L_k$. This means that (see (6) and (4)) $\Phi_i(y) \in E \setminus \{ \emptyset \}$, i.e. $y \in E_i$ (see (7)).

To end the proof it remains now to interpret properly the inclusion (8): either $f(V) = Y \setminus \bigcup_i E_i$ and then the set $f(V)$ is open modulo a first category set in $Y$, or
else, there exists a point \( y \in f(V) \cap E_i \), for some \( i \), and then (see (4), (6) and (7)) \( f^{-1}(y) \cap W_i \) is a nonempty open finite-dimensional set in the fiber \( f^{-1}(y) \).

5. Proof of Theorem 2.1. Let \( X = V_0 \oplus V_1 \) be a Bernstein-like direct sum decomposition as in Theorem 2.1 and let \( V \) be one of the summands of this decomposition.

5.1. Proof of (i). Assume, toward a contradiction, that \( V \) is of first category in \( X \). Then there exists a dense \( G_\delta \)-set \( \mathcal{V} \) in \( X \) disjoint from \( V \) and, by virtue of Lemma 3.1 and property (2) of \( V \), the linear space spanned by this \( G_\delta \)-set has the countable linear dimension. But then, the Baire category theorem easily yields an existence of a nonempty open finite-dimensional subset of \( X \)—a contradiction with infinite-dimensionality of \( X \).

5.2. Proof of (iii). Assume that there exists a homeomorphism of the space \( V \) onto the Cartesian product \( E \times F \) of arbitrary spaces \( E, F \); let \( \tilde{E} \) and \( \tilde{F} \) be completions of \( E \) and \( F \), respectively; and let us extend this homeomorphism, using the Lavrentieff theorem [K1, §31, II], to a homeomorphism \( h : G \rightarrow H \) which maps a \( G_\delta \)-set \( G \) in \( X \) containing \( V \) onto a \( G_\delta \)-set \( H \) in the product \( \tilde{E} \times \tilde{F} \) containing \( E \times F \). Notice that \( h(V) = E \times F \).

Let us assume that the space \( F \) does not contain any nonempty open finite-dimensional sets. Let \( p : \tilde{E} \times \tilde{F} \rightarrow \tilde{E} \) be the projection, let \( Y = p(H) \) and let \( f = p \circ h : G \rightarrow Y \). Let us apply Proposition 4.1 to the map \( f \).

The second possibility in the alternative asserted in Proposition 4.1 cannot occur, as for each \( y \in f(V) = E \) we have \( f^{-1}(y) = h^{-1}((y) \times \tilde{F} \cap H) \), thus \( F \) embeds topologically as a dense subspace in \( f^{-1}(y) \) and hence there is no nonempty open finite-dimensional sets in the space \( f^{-1}(y) \).

Therefore, the first part of this alternative must be true, i.e. \( E \) is open modulo a first category set in \( Y \). Now, \( Y \) being an analytic set, \( Y \) is open modulo a first category set in \( \tilde{E} \) (the Szpilrajn-Marczewski theorem [K1, II, §35]) and hence \( E \) is open modulo a set of first category in \( \tilde{E} \). Moreover, \( E \) is a space of second category, being a factor of the second category space \( V \) (see §5.1), and so \( E \) contains a \( G_\delta \)-subset \( A \) of \( \tilde{E} \) of second category in \( \tilde{E} \). Fix an \( a \in F \) and put \( A' = h^{-1}(A \times \{a\}) \), \( E' = h^{-1}(E \times \{a\}) \). The set \( A' \) is a \( G_\delta \)-set in \( X \) contained in \( V \), hence by Lemma 3.1 and property (2) of \( V \), the linear span of \( A' \) has the countable linear dimension, so \( A' \) is covered by countably many closed in \( X \) finite-dimensional sets \( L_i \). The restriction \( f|E' \) being a homeomorphism onto \( E \), it follows that the closed finite-dimensional subsets \( f(L_i \cap E') \) of \( E \) cover the set \( A \). Finally, the Baire category theorem applied to the completely metrizable space \( A \) of second category in \( E \) yields easily an existence of a nonempty open finite-dimensional subspace of \( E \).

5.3. Proof of (iv). (We use here some improvements of the original reasoning, due to T. Dobrowolski.) Let \( T : V \rightarrow V \) be a linear bounded operator and let \( f \) be the extension of \( T \) to a linear operator mapping the space \( X \) into itself.

If there exists an \( y \in f(X) \backslash V \), then the space \( f^{-1}(y) = x + \ker f \) (\( x \) being any vector with \( f(x) = y \)) is disjoint from \( V \) and hence \( \ker f \) has the countable linear dimension, by Lemma 3.1 and property (2) of \( V \), i.e. \( \ker f \) is finite-dimensional, being a complete space. Thus in this case \( \ker T \subset \ker f \) is finite-dimensional.

The other alternative is that \( f(X) \subset V \). Then \( f(X) \) is an analytic linear subspace of \( V \), and again by Lemma 3.1 and (2), \( f(X) \) has the countable linear dimension. But
the Baire category theorem applied to $X$ yields then easily that $f(X)$ is, in fact, finite-dimensional and so is the range of the operator $T$.

6. Remark. If one is only interested in a construction of a space $V$ with the properties stated in the title, then one can replace the main part of the proof given in §5.2 by the following simpler argument.

6.1. Lemma. Let $Z$ be a compact extension of the space $X$ and let $H$ be a $G_δ$-set in $X \times Z$ containing the square $V \times V$ of the space $V$. Then the set $H^*$ of all $x \in X$ such that the vertical section $H(x)$ of $H$ at $x$ contains a topological copy of the Hilbert cube $I^ω$ is an analytic set.

Proof. Let $(X \times Z) \setminus H = F_1 \cup F_2 \cup \cdots$, $F_1$ being a closed set in $X \times Z$, let $E$ be the space of topological embeddings of $I^ω$ into $Z$ endowed with the compact-open topology, and let $U_n = \{(x, h) \in X \times E : h(I^ω) \cap F_n(x) = \emptyset\}$, $F_n(x)$ being the vertical section of $F_n$ at $x$. Then the sets $U_n$ are open and $H^*$ is the projection of the $G_δ$-set $\bigcap_n U_n$ in $X \times E$ onto the $X$-axis.

Now, let us adopt the notation of §5.2 with $E = F = V$ and $\tilde{E} = \tilde{F} = X$, where we assume also that $X$ is a Banach space.

If $x \not\in V$, then $h^{-1}(\{x\} \times X \cap H)$ is a $G_δ$-set in $X$ disjoint from $V$ and hence it does not contain a topological copy of $I^ω$, by Lemma 3.1 and (2), and so the Hilbert cube does not embed into the vertical section $H(x)$. On the other hand, if $x \in V$, then $H(x)$ is a completely metrizable space containing an infinite-dimensional normed space $V$, and thus the Hilbert cube embeds into $H(x)$, cf. [B-P, Chapter VIII, Theorem 3.1].

It follows that $V$ coincides with the analytic set $H^*$ defined in Lemma 7.1. This however, contradicts Lemma 3.1 and property (2) of the space $V$.

6.2. Remark. One can consider the above reasoning as a reminiscence of an old idea in the theory of analytic sets going back to Mazurkiewicz and Sierpiński, see Kuratowski [K, VII, §43, Proof of Corollary 3].


REFERENCES


