GROUP ACTIONS ON ASPHERICAL $A_k(N)$-MANIFOLDS
WITH NONZERO EULER CHARACTERISTICS

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ABSTRACT. By an aspherical $A_k(N)$-manifold, we mean a compact manifold $M$
together with a map $f$ from $M$ into an aspherical complex $N$ such that $f^*$:
$H^k(N; Q) \to H^k(M; Q)$ is nontrivial. In this paper we study the fixed point set,
degree of symmetry, semisimple degree of symmetry and torus degree of symmetry
of $M$ with the Euler characteristic of $M$ nonzero.

Suppose $M^m$ is a compact connected $m$-manifold. The degree of symmetry $N_f(M)$
(resp. semisimple degree of symmetry $N^s_f(M)$, and torus degree of symmetry $T_t(M)$) is
defined as the supremum of the dimensions of all compact Lie (resp. compact
semisimple Lie, and torus) groups which can act effectively on $M$. Following [10], we
shall say that $M$ is an aspherical $A_k(N)$-manifold, $k$ a nonnegative integer, if $N$ is an
aspherical complex and there exists a continuous map $f: M \to N$ such that $f^*$:
$H^k(N; Q) \to H^k(M; Q)$ is nontrivial. We shall always assume that $\pi_1(N)$ is torsion-
free. For an aspherical $A_k(N)$-manifold $M$, it is known that (cf. [6, 10, 14])
(1) $N_f(M) \le \langle m - k + 1 \rangle$,
and
(2) $N_f(M) \le k + \langle m - k + 1 \rangle$,
where $\langle s \rangle$ denotes $\dim \text{SO}(s)$.

Throughout this paper, the Alexander-Spanier cohomology with compact supports
will be used.

In this paper we shall study the aspherical $A_k(N)$-manifold $M$ with nonzero Euler
characteristic $\chi(X)$. For instance, under this hypothesis, (2) can be improved as
(3) $N_f(M) \le \langle m - k + 1 \rangle$.

The estimate (3) also holds in case $N$ is a Riemannian manifold of negative curvature
(see [6, 10, 14]) if $k > 1$.

THEOREM 1. Let $f: M \to N$ be an aspherical $A_k(N)$-manifold.
(a) If a torus group $G$ acts on $M$ with fixed point set $F$ nonempty, then at least one
component of $F$ is an aspherical cohomology $A_k(N)$-manifold.
(b) If $\chi(X) \neq 0$, then
(4) $T_t(M) \le [(m - k)/2]$.
and
\[ N_T(M) \leq \langle m - k + 1 \rangle, \]
where \([s]\) denotes the largest integer less than or equal to \(s\).

**Proof.** (a) Let \( F = \bigcup_{j=1}^{s} F_j \) and \( f_j = f|F_j; F_j \rightarrow N \). By the Smith theory \([1]\), each \( F_j \) is a cohomology manifold. Suppose that
\[ f_j^*: H^k(N; \mathbb{Q}) \rightarrow H^k(F_j; \mathbb{Q}) \]
is trivial for all \( j = 1, \ldots, s \). We shall get a contradiction. Let \( x \in F \) and \( i: G \rightarrow M \) be the orbit map defined by \( i(g) = gx \). As \( i_*\pi_i(G) = 0 \), it follows from \([6]\) that \( \pi_i(M/G) \cong \pi_i(M)/H \), where \( H \) is a normal subgroup of \( \pi_i(M) \) generated by elements of finite order. Hence if \( \pi: M \rightarrow M/G \) is the natural projection we have
\[ \text{Ker} \left\{ \pi_*: \pi_i(M) \rightarrow \pi_i(M/G) \right\} \subset \text{Ker} \left\{ f_*: \pi_i(M) \rightarrow \pi_i(N) \right\}. \]
Thus, there exists a map \( h: M/G \rightarrow N \) such that \( h\pi \) is homotopic to \( f \) by \([6]\). Let \( q: F \rightarrow M/G \) and \( p: F \rightarrow M \) be inclusions. It is easy to see that \( q^*h^* = p^*\pi^*h^* = p^*f^* = \sum_{j=1}^{s} f^*_j = 0 \) in degree \( k \). Hence if \( z \in H^k(N; \mathbb{Q}) \) be such that \( f^*(z) = z \neq 0 \), then \( q^*h^*(z) = 0 \). From the exact cohomology sequence of the pair \((M/G, F)\), there exists \( z' \in H^k((M - F)/G; \mathbb{Q}) \) such that \( j^*(z') = h^*(z) \), where \( j \) is an inclusion. Consider the following commutative diagram

\[
\begin{array}{ccc}
H^k_G(F; \mathbb{Q}) & \xleftarrow{\sum_{j}^{s} i_j^*} & H^k_G(M; \mathbb{Q}) \\
\downarrow{i^*} & & \downarrow{j^*} \\
H^k(M; \mathbb{Q}) & \xleftarrow{\pi^*} & H^k(M - F; \mathbb{Q})
\end{array}
\]

\[
\begin{array}{ccc}
H^k(G; \mathbb{Q}) & \xleftarrow{j^*} & H^k(M/F; \mathbb{Q}) \\
\downarrow{\pi^*} & & \downarrow{\pi^*} \\
H^k(G; \mathbb{Q}) & \xleftarrow{j^*} & H^k(M/F; \mathbb{Q})
\end{array}
\]

where \( \pi^*_1 \) is induced by natural projection and \( i \) inclusion. It follows that \( \sum_{j=1}^{s} (j^*\pi^*_1 z') = 0 \) and \( i^*(j^*\pi^*_1 z') = z \). Hence \( j^*\pi^*_1 z' \) is \( H^*(B_G; \mathbb{Q}) \)-free. This contradicts to the following Borel Localization Theorem \([7]\):
\[ \sum_{j=1}^{s} S^{-1}i_j^*: S^{-1}H^*_G(M; \mathbb{Q}) \cong S^{-1}H^*_G(F; \mathbb{Q}), \]
where \( S = H^*(B_G; \mathbb{Q}) \) \(-\{0\}\).

(b) Let \( G \) be a torus group acting effectively on \( M \). Since \( \chi(F) = \chi(M) \neq 0 \), the fixed point set \( F \) is not empty. Hence by \([8]\), \( \dim G \leq [(m - \dim F)/2] \) but \( \dim F \geq k \) by (a). The result (4) follows.

To prove (5), let \( G \) be a compact connected Lie group acting effectively on \( M \) and \( \dim G = N_T(M) \). Expressed \( G \) as \( G = (T^s \oplus S)/N \), where \( T^s \) is a torus group of rank \( s \), \( S \) semisimple and \( N \) a finite normal subgroup. According to (1) we have \( \dim S \leq \langle m - k + 1 \rangle \). Let \( r = \text{rank } S \). By (4), we get \( s + r \leq [(m - k)/2] \). Hence
\[ \dim G \leq [(m - k)/2] - r + \dim S. \]
We shall divide the proof into three cases.
Case 1. \( r = 0 \). Then \( \dim G = s \leq \langle (m - k)/2 \rangle \leq \langle m - k + 1 \rangle \).

Case 2. \( r > 0 \), and \( \dim S = \langle m - k + 1 \rangle \). In this case it can be shown (cf. [9]) that \( S \) is locally isomorphic to \( SO(m - k + 1) \), hence

\[
s + r = s + \langle (m - k + 1)/2 \rangle \leq \langle (m - k)/2 \rangle.
\]

It follows that \( s = 0 \).

Case 3. \( r > 0 \), and \( \dim S < \langle m - k + 1 \rangle \).

Let \( S(x) \) be a principal orbit. We know that \( \dim S(x) \leq m - k \) (cf. [6, 10]). Let

\[
W = \begin{cases} 
S(x) & \text{if } \dim S(x) = m - k, \\
S(x) \times S^{m - k - \dim S(x)} & \text{with } S \text{ acting trivially on } S^{m - k - \dim S(x)} \text{ if } \dim S(x) < m - k.
\end{cases}
\]

Consider the action of \( S \) on \( W \). By gap theorem [12], we have

\[
\dim S \leq \langle m - k \rangle + 1
\]

with the following three exceptions:

(i) \( m - k = 4 \), \( W \approx CP^2 \), and \( S \approx SU(3) \);
(ii) \( m - k = 6 \), \( W \approx S^6 \) or \( RP^6 \), and \( S \approx G_2 \);
(iii) \( m - k = 10 \), \( W \approx CP^5 \), and \( S \approx SU(6) \).

It follows from (6) and (7) that

\[
\dim G \leq (m - k)/2 - r + \langle m - k \rangle + 1 \\
\leq (m - k)^2/2 < \langle m - k + 1 \rangle.
\]

This also holds for the three exceptional cases. For instance, (ii) will imply that

\[
\dim G \leq 3 - 2 + 14 < \langle 7 \rangle = 21.
\]

This completes the proof of the theorem.

**Corollary 1.** Let \( M \) be an aspherical \( A_k(N) \)-manifold with \( \chi(M) \neq 0 \).

(a) If \( k = m \), or \( m - 1 \), then \( N_T(M) = 0 \).

(b) If \( k = m - 2 \) and \( \chi(M) \) odd, then \( N_T(M) \leq 1 \).

(c) If \( k = m - 3 \), then \( N_T(M) \leq 3 \).

**Proof.** It suffices to show (b). As in [9], if \( N_T(M) = 3 \), we can show that \( M \) is homeomorphic to \( S^2 \times Z_p \), \( P \) an aspherical \( A_{m-2}(N) \)-manifold. It follows that \( \chi(M) = 2\chi(P) \).

As an immediate corollary to Corollary 1 we have

**Corollary 2.** Let \( M \) be a closed connected oriented manifold and \( M_1 \) a closed aspherical \( A_k(N) \)-manifold and \( \chi(M) + \chi(M_1) \neq 2 \).

(a) If \( k = m \), or \( m - 1 \), then \( N_T(M\#M_1) = 0 \).

(b) If \( k = m - 2 \), and \( \chi(M) + \chi(M_1) \) is odd, then \( N_T(M\#M_1) \leq 1 \).

(c) If \( k = m - 3 \), \( N_T(M\#M_1) \leq 3 \).

Theorem 1(a) was proved in [10] for the case of smooth action and Pontrjagin number of \( f \) nonzero. Corollary 1(a) for \( M = N \) and \( f \) the identity was proved by Conner and Montgomery in [4] (cf. [5]). The special case of Corollary 2(a) for \( M_1 = T^m \) was verified in [3, 9, 13]. Corollary 1(c) is a generalization of [9, Corollary 3.11].
From the proof of Theorem 1(a), we also obtain the following

**Corollary 3.** Let $L$ be a torsion-free group. Let $G$ be a torus group acting on a compact connected manifold $M$ with nonempty fixed point set. Then the projection $\pi: M \to M/G$ induces surjection

$$\pi^*: H^1(M/G; L) \to H^1(M; L).$$

For $L = Q$, this corollary was proved in [2].

For a compact connected manifold $M$ with $\pi_1(M)$ torsion-free, it has a natural $A_k(M)$ structure, where $k$ is the aspherical index $a(M)$ of $M$ which is defined as follows. Let $f: M \to K(\pi_1(M), 1)$ be a classifying map and $a(M)$ be the largest integer $k$ such that $f^*: H^k(K(\pi_1(M), 1); Q) \to H^k(M; Q)$ is nontrivial. For examples, $a(S^{m-k} \times T^k) = k$ and $a(M) = m$ if $M$ is a closed oriented aspherical $m$-manifold.

From (1), (2) and Theorem 1 we have

**Theorem 2.** Let $M$ be a compact connected $m$-manifold. Then

(a) $N^T(M) \leq \left\lfloor m - a(M) + 1 \right\rfloor$.
(b) $N^T(M) \leq m - a(M) + 1$.
(c) If a torus $G$ acts on $M$ with nonempty fixed point set $F$, then at least one component of $F$, say $F_0$, satisfies $H^{a(M)}(F_0; Q) \neq 0$.
(d) If $\chi(M; Q) \neq 0$, then

$$T^T(M) \leq \frac{1}{2}(m - a(M)) \quad \text{and} \quad N^T(M) \leq \left\lfloor m - a(M) + 1 \right\rfloor.$$

Similarly, we also have results corresponding to both Corollary 1 and Corollary 2.

**References**


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